Well-Posed Nonlinear Nonlocal Fracture Models Associated with Double-Well Potentials

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Contents

Abstract

In this chapter, we consider a generic class of bond-based nonlocal nonlinear potentials and formulate the evolution over suitable function spaces. The peridynamic potential considered in this work is a differentiable version of the original bond-based model introduced in Silling (J Mech Phys Solids 48(1):175–209, 2000). The potential associated with the model has two wells where one

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well corresponds to linear elastic behavior and the other corresponds to brittle fracture (see Lipton (J Elast 117(1):21–50, 2014; 124(2):143–191, 2016)). The parameters in the potential can be directly related to the elastic tensor and fracture toughness. In this chapter we show that well-posed formulations of the model can be developed over different function spaces. Here we will consider formulations posed over Hölder spaces and Sobolev spaces. The motivation for the Hölder space formulation is to show a priori convergence for the discrete finite difference method. The motivation for the Sobolev formulation is to show a priori convergence for the finite element method. In the following chapter we will show that the discrete approximations converge to well-posed evolutions. The associated convergence rates are given explicitly in terms of time step and the size of the spatial mesh.

Keywords

Peridynamic modeling · Numerical analysis · Finite difference approximation · Finite element approximation · Stability · Convergence

Introduction

The peridynamic formulation, introduced in Silling (2000), is a nonlocal model for crack propagation in solids. The basic idea is to redefine the strain in terms of the difference quotients of the displacement field and allow for nonlocal forces acting over some finite horizon. This generalized notion of strain allows for the participation of larger class of deformations in the dynamics. The modeling introduces a natural length scale given by the size of the horizon. The force at any given material point is computed by considering the deformation of all neighboring material points contained within the horizon. For linear peridynamic formulations (Silling and Lehoucq, 2008; Emmrich et al., 2013; Mengesha and Du, 2014; Jha and Lipton, 2017c), it is shown that as the nonlocal length scale goes to zero, the peridynamic model collapses to the elastic equilibrium and elastodynamics models. For the nonlinear model introduced in Silling (2000), one may consider a smooth version to find that the energy of the evolution recovers the energy of Linear Elastic Fracture Mechanics as the nonlocal length scale goes to zero (Lipton, 2014, 2016). One of the important points of this model is the fact that as the size of the horizon goes to zero, i.e., when we tend to the local limit, the model behaves as if it is an elastic model away from the crack set. Therefore, in the limit, the model not only converges to linear elasticity in regions with small deformation but also has finite Griffith fracture energy associated with a sharp fracture set. The nonlinear potential can be calibrated so that it gives the same fracture toughness as the Linear Elastic Fracture Mechanics model. Further, the slope of the nonlinear force for small strain is specified precisely by the elastic constant of the material. These results are summarized in Lipton (2016) and Jha and Lipton (2017a). On the other hand to use this model for numerical simulation, we take advantage the regularization given by the nonlocal formulation of the problem. With this in mind we show existence of solutions in more regular spaces for fixed but small horizon and develop a theory for the numerical simulation of fracture problems. In this chapter we present the foundations for the theory and exhibit initial data and boundary conditions so that solutions exist in the Hölder space $C^{0,\gamma}$, $\gamma \in (0, 1]$ and the Sobolev space H^2 (see Jha and Lipton 2017a,b; Diehl et al. 2016). A numerical implementation scheme using the finite difference model is proposed and demonstrated in Lipton et al. (2016). In the following chapters, we show a priori convergence for the finite difference method and finite element method. These results are reported in Jha and Lipton (2017a,b). We show that these discrete approximations converge to the well-posed evolutions described in this chapter. The associated convergence rates are given explicitly in terms of time step and the size of the spatial mesh.

In this chapter we begin by describing bond-based peridynamics and the double-well potential model. Here the nonlocal forces acting between points are given by the derivatives of the potential with respect to the strain (see section "Problem Formulation with Bond-Based Nonlinear Potentials"). The existence of a peridynamic evolution taking values in the space of Hölder continuous functions is presented in section "Existence of Solutions in Hölder Space". The proof uses the Hölder continuity of the nonlocal force with respect to the Hölder norm (see section "Lipschitz Continuity in the Hölder Norm and Existence of a Hölder Continuous Solution"). We then show the existence of a peridynamic evolution in the set of essentially bounded functions taking values in the Sobolev space H^2, the space of functions with function values, and derivatives of order one and two that are square integrable (section "Existence of Solutions in the Sobolev Space H^2"). As before the proof uses the Hölder continuity of the nonlocal force, but now with respect to a norm that is the sum of the H^2 norm and the L^∞ norm, see section "Lipschitz Continuity in the H^2 Norm and Existence of an H^2 Solution". We conclude the chapter observing that the well-posed evolutions over these regular spaces converge to sharp fracture evolutions posed over spaces of functions with jumps (section "Conclusions: Convergence of Regular Solutions in the Limit of Vanishing Horizon").

Problem Formulation with Bond-Based Nonlinear Potentials

Let $D \subset \mathbb{R}^d$, for $d = 2, 3$ be the material domain with characteristic length scale of unity. Every material point $x \in D$ interacts nonlocally with all other material points inside a horizon of length $\epsilon \in (0, 1)$. Let $H_\epsilon(x)$ be the ball of radius ϵ centered at x containing all points y that interact with x. After deformation the material point x assumes position $z = x + u(x)$. In this treatment we assume infinitesimal displacements $u(x)$ so the deformed configuration is the same as the reference configuration and the linearized strain is given by

$$S = S(y, x; u) = \frac{u(y) - u(x)}{|y - x|} \cdot \frac{y - x}{|y - x|}.$$

We let t denote time and the displacement field $u(t,x)$ evolves according to Newton's second law

$$\rho\partial_{tt}^2 u(t,x) = -\nabla PD^\epsilon(u(t))(x) + b(t,x) \tag{1}$$

for all $x \in D$. Here the body force applied to the domain D can evolve with time and is denoted by $b(t,x)$. Without loss of generality, we will assume $\rho = 1$. The peridynamic force denoted by $-\nabla PD^\epsilon(u)(x)$ is given by summing up all forces acting on x

$$-\nabla PD^\epsilon(u)(x) = \frac{2}{\epsilon^d \omega_d}\int_{H_\epsilon(x)} \partial_S W^\epsilon(S, y-x)\frac{y-x}{|y-x|}dy,$$

where $\partial_S W^\epsilon$ is the force exerted on x by y and is given by the derivative of the nonlocal two-point potential $W^\epsilon(S, y-x)$ with respect to the strain and ω_d is volume of unit ball in dimension d.

Let ∂D be the boundary of material domain D. The Dirichlet boundary condition on u is

$$u(t,x) = 0 \qquad \forall x \in \partial D, \forall t \in [0,T] \tag{2}$$

and initial condition is

$$u(0,x) = u_0(x) \qquad \text{and} \qquad \partial_t u(0,x) = v_0(x). \tag{3}$$

The initial data and solution $u(t,x)$ are extended by 0 outside D.

The total energy $\mathcal{E}^\epsilon(u)(t)$ is given by the sum of kinetic and potential energy given by

$$\mathcal{E}^\epsilon(u)(t) = \frac{1}{2}||\dot{u}(t)||_{L^2} + PD^\epsilon(u(t)), \tag{4}$$

where potential energy PD^ϵ is given by

$$PD^\epsilon(u) = \frac{1}{2}\int_D \left[\frac{1}{\epsilon^d \omega_d}\int_{H_\epsilon(x)} W^\epsilon(S(u), y-x)dy\right]dx.$$

Nonlocal Potential

We consider potentials W^ϵ of the form

$$W^\epsilon(S, y-x) = \omega(x)\omega(y)\frac{J^\epsilon(|y-x|)}{\epsilon} f(|y-x|S^2), \tag{5}$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is assumed to be positive, smooth, and concave with the following properties

$$\lim_{r\to 0^+} \frac{f(r)}{r} = f'(0), \qquad \lim_{r\to\infty} f(r) = f_\infty < \infty. \tag{6}$$

The peridynamic force $-\nabla PD^\epsilon$ is written as

$$-\nabla PD^\epsilon(u)(x) = \frac{4}{\epsilon^{d+1}\omega_d}\int_{H_\epsilon(x)} \omega(x)\omega(y)J^\epsilon(|y-x|)f'(|y-x|S(u)^2)S(u)e_{y-x}dy, \tag{7}$$

where we write $S(u) = S(y, x; u)$ and $e_{y-x} = \frac{y-x}{|y-x|}$.

The function $J^\epsilon(|y-x|)$ models the influence of separation between points y and x. Here $J^\epsilon(|y-x|) = J(|y-x|/\epsilon)$, and we define J to be zero outside the ball $\{\xi : |\xi| < 1\} = H_1(0)$ and $0 \le J(|\xi|) \le M$ for all $\xi \in H_1(0)$.

The boundary function $\omega(x)$ is nonnegative and takes the value 1 for points x inside D of distance ϵ away from the boundary ∂D. Inside the boundary layer of width ϵ, the function $\omega(x)$ smoothly decreases from 1 to 0 taking the value 0 on ∂D.

The potential described in Eq. 5 gives the convex-concave dependence (see Fig. 1) of $W(S, y-x)$ on the strain S for fixed $y-x$. Here the potential has a well at zero strain and has a second well at infinite strain given by the horizontal asymptote. Initially the deformation is elastic for small strains and then softens as the strain becomes large; this is illustrated in Fig. 2. The critical strain where the force between x and y begins to soften is given by $S_c(y, x) := \bar{r}/\sqrt{|y-x|}$, and the force decreases monotonically for

$$|S(y, x; u)| > S_c. \tag{8}$$

Here $\bar{r}$ is the inflection point of $r \mapsto f(r^2)$ and is the root of the following equation:

$$f'(r^2) + 2r^2 f''(r^2) = 0. \tag{9}$$

In (Theorem 5.2, Lipton 2016), it is shown that in the limit $\epsilon \to 0$, the peridynamic solution has bounded linear elastic fracture energy, provided the initial data (u_0, v_0) has bounded linear elastic fracture energy and u_0 is bounded. The elastic constants (Lamé constant λ and μ) and energy release rate of the limiting energy are given by

$$\lambda = \mu = C_d f'(0) M_d, \quad \mathcal{G}_c = \frac{2\omega_{d-1}}{\omega_d} f_\infty M_d$$

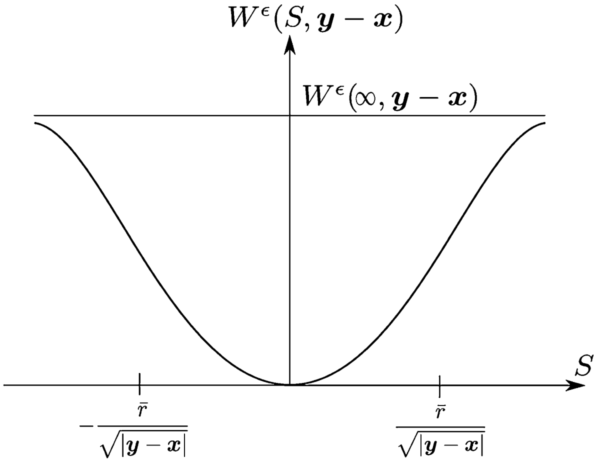

Fig. 1 Two-point potential $W^\epsilon(S, y-x)$ as a function of strain S for fixed $y-x$

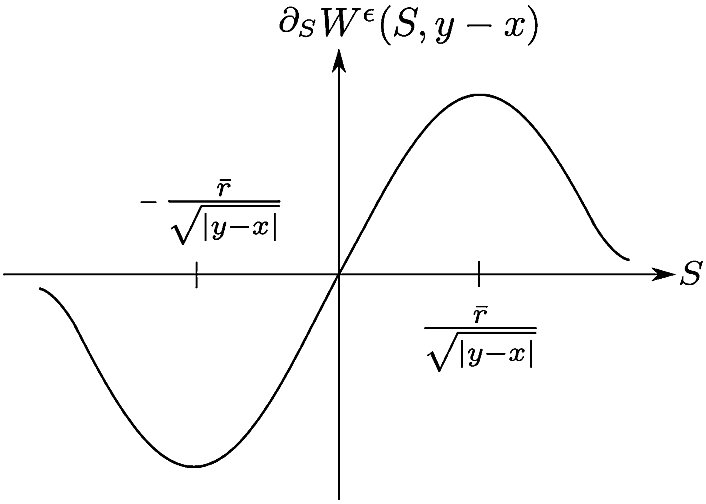

Fig. 2 Nonlocal force $\partial_S W^\epsilon(S, y-x)$ as a function of strain S for fixed $y-x$. Second derivative of $W^\epsilon(S, y-x)$ is zero at $\pm\bar{r}/\sqrt{|y-x|}$

where $M_d = \int_0^1 J(r)r^d dr$ and $f_\infty = \lim_{r\to\infty} f(r)$. $C_d = 2/3, 1/4, 1/5$ for $d = 1, 2, 3$, respectively, and $\omega_n = 1, 2, \pi, 4\pi/3$ for $n = 0, 1, 2, 3$. Therefore, $f'(0)$ and f_∞ are determined by the Lamé constant λ and fracture toughness $\mathcal{G}_c$.

Weak Formulation

We now give the weak formulation of the evolution. Multiplying Eq. 1 by a smooth test function $\tilde{u}$ with $\tilde{u} = 0$ on ∂D, we get

$$(\ddot{u}(t), \tilde{u}) = (-\nabla PD^\epsilon(u(t)), \tilde{u}) + (b(t), \tilde{u}).$$

We denote L^2 dot product of u, v as (u, v). An integration by parts easily shows for all smooth u, v taking zero boundary values that

$$(-\nabla PD^\epsilon(u), v) = -a^\epsilon(u, v),$$

where

$$\begin{aligned} &a^\epsilon(u, v) \\ &= \frac{2}{\epsilon^{d+1}\omega_d} \int_D \int_{H_\epsilon(x)} \omega(x)\omega(y) J^\epsilon(|y-x|) \\ &\quad f'(|y-x|S(u)^2)|y-x|S(u)S(v)dydx. \end{aligned} \tag{10}$$

Finally, the weak form of the evolution in terms of operator a^ϵ becomes

$$(\ddot{u}(t), \tilde{u}) + a^\epsilon(u(t), \tilde{u}) = (b(t), \tilde{u}). \tag{11}$$

Using definition of a^ϵ in Eq. 10, one easily sees that

$$\frac{d}{dt}\mathcal{E}^\epsilon(u)(t) = (\ddot{u}(t), \dot{u}(t)) + a^\epsilon(u(t), \dot{u}(t)). \tag{12}$$

In the sequel the notation $||\cdot||$ denotes the L^2 norm on D, and $||\cdot||_\infty$ is used for the L^∞ norm on D and $||\cdot||_2$ for Sobolev H^2 norm on D.

Existence of Solutions in Hölder Space

In this section, we establish the existence of solutions in Hölder space. Here we follow the approach developed in Jha and Lipton (2017a). Let $C^{0,\gamma}(D; \mathbb{R}^d)$ be the Hölder space with exponent $\gamma \in (0, 1]$. The closure of continuous functions with compact support on D in the supremum norm is denoted by $C_0(D)$. We identify functions in $C_0(D)$ with their unique continuous extensions to $\overline{D}$. It is easily seen that functions belonging to this space take the value zero on the boundary of D (see, e.g., Driver 2003). We introduce $C_0^{0,\gamma}(D) = C^{0,\gamma}(D) \cap C_0(D)$. Here we extend all functions in $C_0^{0,\gamma}(D)$ by zero outside D. The norm of $\boldsymbol{u} \in C_0^{0,\gamma}(D; \mathbb{R}^d)$ is taken to be

$$\|\boldsymbol{u}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} := \sup_{\boldsymbol{x}\in D} |\boldsymbol{u}(\boldsymbol{x})| + [\boldsymbol{u}]_{C^{0,\gamma}(D;\mathbb{R}^d)},$$

where $[\boldsymbol{u}]_{C^{0,\gamma}(D;\mathbb{R}^d)}$ is the Hölder semi norm and given by

$$[\boldsymbol{u}]_{C^{0,\gamma}(D;\mathbb{R}^d)} := \sup_{\substack{\boldsymbol{x}\neq\boldsymbol{y},\\ \boldsymbol{x},\boldsymbol{y}\in D}} \frac{|\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{\gamma}},$$

and $C_0^{0,\gamma}(D;\mathbb{R}^d)$ is a Banach space with this norm. Here we make the hypothesis that the domain function ω belongs to $C_0^{0,\gamma}(D;\mathbb{R}^d)$.

We write the evolution Eq. 1 as an equivalent first-order system with $y_1(t) = \boldsymbol{u}(t)$ and $y_2(t) = \boldsymbol{v}(t)$ with $\boldsymbol{v}(t) = \partial_t \boldsymbol{u}(t)$. Let $y = (y_1, y_2)^T$ where $y_1, y_2 \in C_0^{0,\gamma}(D;\mathbb{R}^d)$ and let $F^\epsilon(y,t) = (F_1^\epsilon(y,t), F_2^\epsilon(y,t))^T$ such that

$$F_1^\epsilon(y,t) := y_2 \tag{13}$$

$$F_2^\epsilon(y,t) := -\nabla PD^\epsilon(y_1) + \boldsymbol{b}(t). \tag{14}$$

The initial boundary value associated with the evolution Eq. 1 is equivalent to the initial boundary value problem for the first-order system given by

$$\frac{d}{dt} y = F^\epsilon(y,t), \tag{15}$$

with initial condition given by $y(0) = (\boldsymbol{u}_0, \boldsymbol{v}_0)^T \in C_0^{0,\gamma}(D;\mathbb{R}^d) \times C_0^{0,\gamma}(D;\mathbb{R}^d)$.

The function $F^\epsilon(y,t)$ satisfies the Lipschitz continuity given by the following theorem.

Proposition 1 (Lipschitz continuity and bound). *Let $X = C_0^{0,\gamma}(D;\mathbb{R}^d) \times C_0^{0,\gamma}(D;\mathbb{R}^d)$. The function $F^\epsilon(y,t) = (F_1^\epsilon, F_2^\epsilon)^T$, as defined in Eqs. 13 and 14, is Lipschitz continuous in any bounded subset of X. We have, for any $y, z \in X$ and $t > 0$,*

$$\begin{aligned} &\|F^\epsilon(y,t) - F^\epsilon(z,t)\|_X \\ &\leq \frac{\left(L_1 + L_2\left(\|\omega\|_{C^{0,\gamma}(D)} + \|y\|_X + \|z\|_X\right)\right)}{\epsilon^{2+\alpha(\gamma)}} \|y - z\|_X \end{aligned} \tag{16}$$

where L_1, L_2 are independent of $\boldsymbol{u}, \boldsymbol{v}$ and depend on peridynamic potential function f and influence function J and the exponent $\alpha(\gamma)$ is given by

$$\alpha(\gamma) = \begin{cases} 0 & \text{if } \gamma \geq 1/2 \\ 1/2 - \gamma & \text{if } \gamma < 1/2. \end{cases}$$

Furthermore for any $y \in X$ and any $t \in [0,T]$, we have the bound

$$\|F^\epsilon(y,t)\|_X \leq \frac{L_3}{\epsilon^{2+\alpha(\gamma)}}(1 + \|\omega\|_{C^{0,\gamma}(D)} + \|y\|_X) + b \tag{17}$$

where $b = \sup_t \|\boldsymbol{b}(t)\|_{C^{0,\gamma}(D;\mathbb{R}^d)}$ *and* L_3 *is independent of* y.

We easily see that on choosing $z = 0$ in Eq. 16 that $-\nabla PD^\epsilon(\boldsymbol{u})(\boldsymbol{x})$ is in $C^{0,\gamma}(D;\mathbb{R}^3)$ provided that $\boldsymbol{u}$ belongs to $C^{0,\gamma}(D;\mathbb{R}^3)$. Since $-\nabla PD^\epsilon(\boldsymbol{u})(\boldsymbol{x})$ takes the value 0 on ∂D, we conclude that $-\nabla PD^\epsilon(\boldsymbol{u})(\boldsymbol{x})$ belongs to $C_0^{0,\gamma}(D;\mathbb{R}^3)$.

The following theorem gives the existence and uniqueness of solution in any given time domain $I_0 = (-T, T)$.

Theorem 1 (Existence and uniqueness of Hölder solutions of cohesive dynamics over finite time intervals). *For any initial condition* $x_0 \in X = C_0^{0,\gamma}(D;\mathbb{R}^d) \times C_0^{0,\gamma}(D;\mathbb{R}^d)$, *time interval* $I_0 = (-T, T)$, *and right-hand side* $\boldsymbol{b}(t)$ *continuous in time for* $t \in I_0$ *such that* $\boldsymbol{b}(t)$ *satisfies* $\sup_{t\in I_0} ||\boldsymbol{b}(t)||_{C^{0,\gamma}} < \infty$, *there is a unique solution* $y(t) \in C^1(I_0; X)$ *of*

$$y(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau)\, d\tau,$$

or equivalently

$$y'(t) = F^\epsilon(y(t), t), \textit{with } y(0) = x_0,$$

where $y(t)$ *and* $y'(t)$ *are Lipschitz continuous in time for* $t \in I_0$.

The proof of this theorem is given in the following section.

Lipschitz Continuity in the Hölder Norm and Existence of a Hölder Continuous Solution

In this section, we prove Proposition 1.

Proof of Proposition 1

Let $I = [0, T]$ be the time domain and $X = C_0^{0,\gamma}(D;\mathbb{R}^d) \times C_0^{0,\gamma}(D;\mathbb{R}^d)$. Recall that $F^\epsilon(y,t) = (F_1^\epsilon(y,t), F_2^\epsilon(y,t))$, where $F_1^\epsilon(y,t) = y^2$ and $F_2^\epsilon(y,t) = -\boldsymbol{\nabla} PD^\epsilon(y^1) + \boldsymbol{b}(t)$. Given $t \in I$ and $y = (y^1, y^2), z = (z^1, z^2) \in X$, we have

$$\begin{aligned} &\|F^\epsilon(y,t) - F^\epsilon(z,t)\|_X \\ &\leq \left\|y^2 - z^2\right\|_{C^{0,\gamma}(D;\mathbb{R}^d)} + \left\|-\boldsymbol{\nabla} PD^\epsilon(y^1) + \boldsymbol{\nabla} PD^\epsilon(z^1)\right\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \end{aligned} \tag{18}$$

Therefore, to prove the Eq. 16, we only need to analyze the second term in above inequality. Let $\boldsymbol{u}, \boldsymbol{v} \in C_0^{0,\gamma}(D;\mathbb{R}^d)$, then we have

$$
\begin{aligned}
&\|-\nabla PD^\epsilon(\boldsymbol{u}) - (-\nabla PD^\epsilon(\boldsymbol{v}))\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \\
&= \sup_{\boldsymbol{x}\in D} |-\nabla PD^\epsilon(\boldsymbol{u})(\boldsymbol{x}) - (-\nabla PD^\epsilon(\boldsymbol{v})(\boldsymbol{x}))| \\
&\quad + \sup_{\substack{\boldsymbol{x}\neq \boldsymbol{y},\\ \boldsymbol{x},\boldsymbol{y}\in D}} \frac{|(-\nabla PD^\epsilon(\boldsymbol{u}) + \nabla PD^\epsilon(\boldsymbol{v}))(\boldsymbol{x}) - (-\nabla PD^\epsilon(\boldsymbol{u}) + \nabla PD^\epsilon(\boldsymbol{v}))(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}.
\end{aligned}
\tag{19}
$$

Note that the force $-\nabla PD^\epsilon(\boldsymbol{u})(\boldsymbol{x})$ can be written as follows:

$$
\begin{aligned}
&-\nabla PD^\epsilon(\boldsymbol{u})(\boldsymbol{x}) \\
&= \frac{4}{\epsilon^{d+1}\omega_d} \int_{H_\epsilon(\boldsymbol{x})} \omega(\boldsymbol{x})\omega(\boldsymbol{y}) J(\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\epsilon}) f'(|\boldsymbol{y}-\boldsymbol{x}|\, S(\boldsymbol{y},\boldsymbol{x};\boldsymbol{u})^2) S(\boldsymbol{y},\boldsymbol{x};\boldsymbol{u}) \frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|} d\boldsymbol{y} \\
&= \frac{4}{\epsilon\omega_d} \int_{H_1(\boldsymbol{0})} \omega(\boldsymbol{x})\omega(\boldsymbol{x}+\epsilon\boldsymbol{\xi}) J(|\boldsymbol{\xi}|) f'(\epsilon\, |\boldsymbol{\xi}|\, S(\boldsymbol{x}+\epsilon\boldsymbol{\xi},\boldsymbol{x};\boldsymbol{u})^2) S(\boldsymbol{x}+\epsilon\boldsymbol{\xi},\boldsymbol{x};\boldsymbol{u}) \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} d\boldsymbol{\xi}.
\end{aligned}
$$

where we substituted $\partial_S W^\epsilon$ using Eq. 5. In the second step, we introduced the change in variable $\boldsymbol{y} = \boldsymbol{x} + \epsilon\boldsymbol{\xi}$.

Let $F_1 : \mathbb{R} \to \mathbb{R}$ be defined as $F_1(S) = f(S^2)$. Then $F_1'(S) = f'(S^2)2S$. Using the definition of F_1, we have

$$
2Sf'(\epsilon\, |\boldsymbol{\xi}|\, S^2) = \frac{F_1'(\sqrt{\epsilon\, |\boldsymbol{\xi}|}S)}{\sqrt{\epsilon\, |\boldsymbol{\xi}|}}.
$$

Because f is assumed to be positive, smooth, and concave and is bounded far away, we have the following bound on derivatives of F_1

$$
\sup_r |F_1'(r)| = F_1'(\bar{r}) =: C_1 \tag{20}
$$

$$
\sup_r |F_1''(r)| = \max\{F_1''(0), F_1''(\hat{u})\} =: C_2 \tag{21}
$$

$$
\sup_r |F_1'''(r)| = \max\{F_1'''(\bar{u}_2), F_1'''(\tilde{u}_2)\} =: C_3. \tag{22}
$$

where $\bar{r}$ is the inflection point of $f(r^2)$, i.e., $F_1''(\bar{r}) = 0$. $\{0, \hat{u}\}$ are the maxima of $F_1''(r)$. $\{\bar{u}, \tilde{u}\}$ are the maxima of $F_1'''(r)$. By chain rule and by considering the assumption on f, we can show that $\bar{r}, \hat{u}, \bar{u}_2, \tilde{u}_2$ exists and the C_1, C_2, C_3 are bounded. Figures 3, 4, and 5 show the generic graphs of $F_1'(r)$, $F_1''(r)$, and $F_1'''(r)$, respectively.

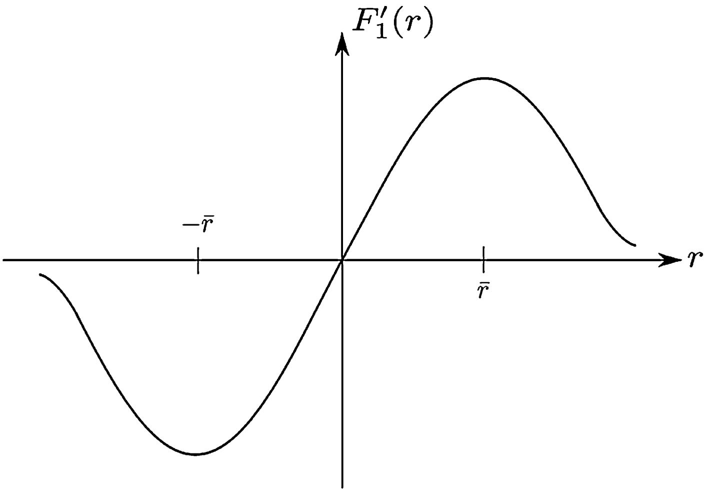

Fig. 3 Generic plot of $F_1'(r)$. $|F_1'(r)|$ is bounded by $\left|F_1'(\bar{r})\right|$

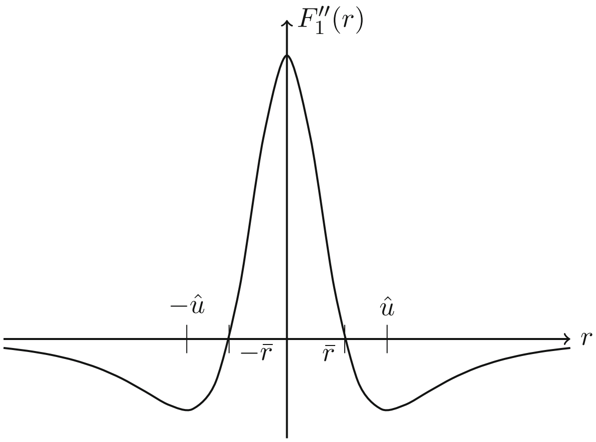

Fig. 4 Generic plot of $F_1''(r)$. At $\pm\bar{r}$, $F_1''(r) = 0$. At $\pm\hat{u}$, $F_1'''(r) = 0$

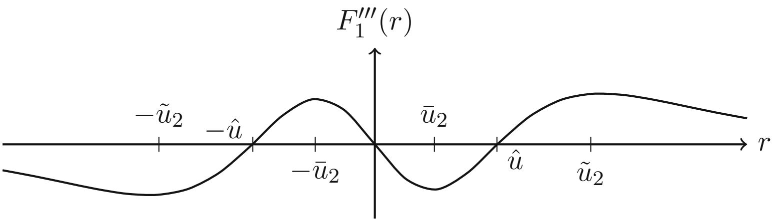

Fig. 5 Generic plot of $F_1'''(r)$. At $\pm\bar{u}_2$ and $\pm\tilde{u}_2$, $F_1'''' = 0$

The nonlocal force $-\nabla PD^\epsilon$ can be written as

$$
\begin{aligned}
&-\nabla PD^\epsilon(\boldsymbol{u})(\boldsymbol{x}) \\
&= \frac{2}{\epsilon\omega_d}\int_{H_1(\boldsymbol{0})}\omega(\boldsymbol{x})\omega(\boldsymbol{x}+\epsilon\boldsymbol{\xi})J(|\boldsymbol{\xi}|)F_1'(\sqrt{\epsilon|\boldsymbol{\xi}|}S(\boldsymbol{x}+\epsilon\boldsymbol{\xi},\boldsymbol{x};\boldsymbol{u}))\frac{1}{\sqrt{\epsilon|\boldsymbol{\xi}|}}\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}d\boldsymbol{\xi}.
\end{aligned}
\tag{23}
$$

To simplify the calculations, we use following notation:

$$
\begin{aligned}
\bar{\boldsymbol{u}}(\boldsymbol{x}) &:= \boldsymbol{u}(\boldsymbol{x}+\epsilon\boldsymbol{\xi})-\boldsymbol{u}(\boldsymbol{x}),\\
\bar{\boldsymbol{u}}(\boldsymbol{y}) &:= \boldsymbol{u}(\boldsymbol{y}+\epsilon\boldsymbol{\xi})-\boldsymbol{u}(\boldsymbol{y}),\\
(\boldsymbol{u}-\boldsymbol{v})(\boldsymbol{x}) &:= \boldsymbol{u}(\boldsymbol{x})-\boldsymbol{v}(\boldsymbol{x}),
\end{aligned}
$$

and $\overline{(\boldsymbol{u}-\boldsymbol{v})}(\boldsymbol{x})$ is defined similar to $\bar{\boldsymbol{u}}(\boldsymbol{x})$. Also, let

$$
s=\epsilon|\boldsymbol{\xi}|,\quad \boldsymbol{e}=\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}.
$$

In what follows, we will come across the integral of type $\int_{H_1(\boldsymbol{0})}J(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^{-\alpha}d\boldsymbol{\xi}$. Recall that $0\le J(|\boldsymbol{\xi}|)\le M$ for all $\boldsymbol{\xi}\in H_1(\boldsymbol{0})$ and $J(|\boldsymbol{\xi}|)=0$ for $\boldsymbol{\xi}\notin H_1(\boldsymbol{0})$. Therefore, let

$$
\bar{J}_\alpha := \frac{1}{\omega_d}\int_{H_1(\boldsymbol{0})}J(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^{-\alpha}d\boldsymbol{\xi}. \tag{24}
$$

With notations above, we note that $S(\boldsymbol{x}+\epsilon\boldsymbol{\xi},\boldsymbol{x};\boldsymbol{u})=\bar{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{e}/s$. $-\nabla PD^\epsilon$ can be written as

$$
-\nabla PD^\epsilon(\boldsymbol{u})(\boldsymbol{x}) = \frac{2}{\epsilon\omega_d}\int_{H_1(\boldsymbol{0})}\omega(\boldsymbol{x})\omega(\boldsymbol{x}+\epsilon\boldsymbol{\xi})J(|\boldsymbol{\xi}|)F_1'(\bar{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{e}/\sqrt{s})\frac{1}{\sqrt{s}}\boldsymbol{e}\,d\boldsymbol{\xi}. \tag{25}
$$

We first estimate the term $|-\nabla PD^\epsilon(\boldsymbol{u})(\boldsymbol{x})-(-\nabla PD^\epsilon(\boldsymbol{v})(\boldsymbol{x}))|$ in Eq. 19.

$$
\begin{aligned}
&|-\nabla PD^\epsilon(\boldsymbol{u})(\boldsymbol{x})-(-\nabla PD^\epsilon(\boldsymbol{v})(\boldsymbol{x}))|\\
&\le \left|\frac{2}{\epsilon\omega_d}\int_{H_1(\boldsymbol{0})}\omega(\boldsymbol{x})\omega(\boldsymbol{x}+\epsilon\boldsymbol{\xi})J(|\boldsymbol{\xi}|)\frac{\left(F_1'(\bar{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{e}/\sqrt{s})-F_1'(\bar{\boldsymbol{v}}(\boldsymbol{x})\cdot\boldsymbol{e}/\sqrt{s})\right)}{\sqrt{s}}\boldsymbol{e}\,d\boldsymbol{\xi}\right|\\
&\le \left|\frac{2}{\epsilon\omega_d}\int_{H_1(\boldsymbol{0})}J(|\boldsymbol{\xi}|)\frac{1}{\sqrt{s}}\left|F_1'(\bar{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{e}/\sqrt{s})-F_1'(\bar{\boldsymbol{v}}(\boldsymbol{x})\cdot\boldsymbol{e}/\sqrt{s})\right|d\boldsymbol{\xi}\right|
\end{aligned}
$$

$$\leq \sup_r \left|F_1''(r)\right| \left|\frac{2}{\epsilon\omega_d}\int_{H_1(\mathbf{0})} J(|\boldsymbol{\xi}|)\frac{1}{\sqrt{s}}\left|\bar{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{e}/\sqrt{s} - \bar{\boldsymbol{v}}(\boldsymbol{x})\cdot\boldsymbol{e}/\sqrt{s}\right| d\boldsymbol{\xi}\right|$$

$$\leq \frac{2C_2}{\epsilon\omega_d}\left|\int_{H_1(\mathbf{0})} J(|\boldsymbol{\xi}|)\frac{|\bar{\boldsymbol{u}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{x})|}{\epsilon\,|\boldsymbol{\xi}|} d\boldsymbol{\xi}\right|. \tag{26}$$

Here we have used the fact that $|\omega(\boldsymbol{x})| \leq 1$ and for a vector $\boldsymbol{e}$ such that $|\boldsymbol{e}| = 1$, $|\boldsymbol{a}\cdot\boldsymbol{e}| \leq |\boldsymbol{a}|$ holds and $|\alpha\boldsymbol{e}| \leq |\alpha|$ holds for all $\boldsymbol{a} \in \mathbb{R}^d, \alpha \in \mathbb{R}$. Using the fact that $\boldsymbol{u}, \boldsymbol{v} \in C_0^{0,\gamma}(D;\mathbb{R}^d)$, we have

$$\begin{aligned}\frac{|\bar{\boldsymbol{u}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{x})|}{s} &= \frac{|(\boldsymbol{u}-\boldsymbol{v})(\boldsymbol{x}+\epsilon\boldsymbol{\xi}) - (\boldsymbol{u}-\boldsymbol{v})(\boldsymbol{x})|}{(\epsilon\,|\boldsymbol{\xi}|)^\gamma}\frac{1}{(\epsilon\,|\boldsymbol{\xi}|)^{1-\gamma}}\\ &\leq \|\boldsymbol{u}-\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}\frac{1}{(\epsilon\,|\boldsymbol{\xi}|)^{1-\gamma}}.\end{aligned}$$

Substituting the estimate given above, we get

$$|-\nabla PD^\epsilon(\boldsymbol{u})(\boldsymbol{x}) - (-\nabla PD^\epsilon(\boldsymbol{v})(\boldsymbol{x}))| \leq \frac{2C_2\bar{J}_{1-\gamma}}{\epsilon^{2-\gamma}}\|\boldsymbol{u}-\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}, \tag{27}$$

where C_2 is given by Eq. 21 and $\bar{J}_{1-\gamma}$ is given by Eq. 24.

We now estimate the second term in Eq. 19. To simplify notation, we write $\tilde{\omega}(\boldsymbol{x},\boldsymbol{\xi}) = \omega(\boldsymbol{x})\omega(\boldsymbol{x}+\epsilon\boldsymbol{\xi})$ and with the help of Eq. 25, we get

$$\begin{aligned}&\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}\,|(-\nabla PD^\epsilon(\boldsymbol{u}) + \nabla PD^\epsilon(\boldsymbol{v}))(\boldsymbol{x}) - (-\nabla PD^\epsilon(\boldsymbol{u}) + \nabla PD^\epsilon(\boldsymbol{v}))(\boldsymbol{y})|\\
&= \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}\left|\frac{2}{\epsilon\omega_d}\int_{H_1(0)} J(|\boldsymbol{\xi}|)\frac{1}{\sqrt{s}}\times\left(\tilde{\omega}(\boldsymbol{x},\boldsymbol{\xi})(F_1'(\frac{\bar{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{e}}{\sqrt{s}}) - F_1'(\frac{\bar{\boldsymbol{v}}(\boldsymbol{x}))\cdot\boldsymbol{e}}{\sqrt{s}}))\right.\right.\\
&\left.\left.-\tilde{\omega}(\boldsymbol{y},\boldsymbol{\xi})(F_1'(\frac{\bar{\boldsymbol{u}}(\boldsymbol{y})\cdot\boldsymbol{e}}{\sqrt{s}}) - F_1'(\frac{\bar{\boldsymbol{v}}(\boldsymbol{y}))\cdot\boldsymbol{e}}{\sqrt{s}})\right)\boldsymbol{e}\,d\boldsymbol{\xi}\right|\\
&\leq \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}\left|\frac{2}{\epsilon\omega_d}\right|\int_{H_1(0)} J(|\boldsymbol{\xi}|)\frac{1}{\sqrt{s}}\times\\
&\quad|\tilde{\omega}(\boldsymbol{x},\boldsymbol{\xi})(F_1'(\frac{\bar{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{e}}{\sqrt{s}}) - F_1'(\frac{\bar{\boldsymbol{v}}(\boldsymbol{x})\cdot\boldsymbol{e}}{\sqrt{s}})) - \tilde{\omega}(\boldsymbol{y},\boldsymbol{\xi})(F_1'(\frac{\bar{\boldsymbol{u}}(\boldsymbol{y})\cdot\boldsymbol{e}}{\sqrt{s}})\\
&\quad - F_1'(\frac{\bar{\boldsymbol{v}}(\boldsymbol{y})\cdot\boldsymbol{e}}{\sqrt{s}}))|d\boldsymbol{\xi}.\end{aligned} \tag{28}$$

We analyze the integrand in above equation. We let H be defined by

$$H := \frac{|\tilde{\omega}(\boldsymbol{x},\boldsymbol{\xi})(F_1'(\frac{\bar{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{e}}{\sqrt{s}}) - F_1'(\frac{\bar{\boldsymbol{v}}(\boldsymbol{x})\cdot\boldsymbol{e}}{\sqrt{s}})) - \tilde{\omega}(\boldsymbol{y},\boldsymbol{\xi})(F_1'(\frac{\bar{\boldsymbol{u}}(\boldsymbol{y})\cdot\boldsymbol{e}}{\sqrt{s}}) - F_1'(\frac{\bar{\boldsymbol{v}}(\boldsymbol{y})\cdot\boldsymbol{e}}{\sqrt{s}}))|}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}.$$

Let $\boldsymbol{r} : [0,1] \times D \to \mathbb{R}^d$ be defined as

$$\boldsymbol{r}(l, \boldsymbol{x}) = \bar{\boldsymbol{v}}(\boldsymbol{x}) + l(\bar{\boldsymbol{u}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{x})).$$

Note $\partial \boldsymbol{r}(l, \boldsymbol{x})/\partial l = \bar{\boldsymbol{u}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{x})$. Using $\boldsymbol{r}(l, \boldsymbol{x})$, we have

$$F_1'(\bar{\boldsymbol{u}}(\boldsymbol{x}) \cdot \boldsymbol{e}/\sqrt{s}) - F_1'(\bar{\boldsymbol{v}}(\boldsymbol{x}) \cdot \boldsymbol{e}/\sqrt{s}) = \int_0^1 \frac{\partial F_1'(\boldsymbol{r}(l, \boldsymbol{x}) \cdot \boldsymbol{e}/\sqrt{s})}{\partial l} dl \tag{29}$$

$$= \int_0^1 \frac{\partial F_1'(\boldsymbol{r} \cdot \boldsymbol{e}/\sqrt{s})}{\partial \boldsymbol{r}}\Big|_{\boldsymbol{r}=\boldsymbol{r}(l,\boldsymbol{x})} \cdot \frac{\partial \boldsymbol{r}(l, \boldsymbol{x})}{\partial l} dl. \tag{30}$$

Similarly, we have

$$F_1'(\bar{\boldsymbol{u}}(\boldsymbol{y}) \cdot \boldsymbol{e}/\sqrt{s}) - F_1'(\bar{\boldsymbol{v}}(\boldsymbol{y}) \cdot \boldsymbol{e}/\sqrt{s}) = \int_0^1 \frac{\partial F_1'(\boldsymbol{r} \cdot \boldsymbol{e}/\sqrt{s})}{\partial \boldsymbol{r}}\Big|_{\boldsymbol{r}=\boldsymbol{r}(l,\boldsymbol{y})} \cdot \frac{\partial \boldsymbol{r}(l, \boldsymbol{y})}{\partial l} dl. \tag{31}$$

Note that

$$\frac{\partial F_1'(\boldsymbol{r} \cdot \boldsymbol{e}/\sqrt{s})}{\partial \boldsymbol{r}}\Big|_{\boldsymbol{r}=\boldsymbol{r}(l,\boldsymbol{y})} = F_1''(\boldsymbol{r}(l, \boldsymbol{x}) \cdot \boldsymbol{e}/\sqrt{s}) \frac{\boldsymbol{e}}{\sqrt{s}}. \tag{32}$$

Combining Eqs. 30, 31, and 32 gives

$$\begin{aligned} H &= \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^\gamma} \Big| \int_0^1 \big(\tilde{\omega}(\boldsymbol{x}, \boldsymbol{\xi}) F_1''(\boldsymbol{r}(l, \boldsymbol{x}) \cdot \boldsymbol{e}/\sqrt{s})(\bar{\boldsymbol{u}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{x})) \\ &\qquad - \tilde{\omega}(\boldsymbol{y}, \boldsymbol{\xi}) F_1''(\boldsymbol{r}(l, \boldsymbol{y}) \cdot \boldsymbol{e}/\sqrt{s})(\bar{\boldsymbol{u}}(\boldsymbol{y}) - \bar{\boldsymbol{v}}(\boldsymbol{y}))\big) \cdot \frac{\boldsymbol{e}}{\sqrt{s}} dl \Big| \\ &\le \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^\gamma} \frac{1}{\sqrt{s}} \Big| \int_0^1 |\tilde{\omega}(\boldsymbol{x}, \boldsymbol{\xi}) F_1''(\boldsymbol{r}(l, \boldsymbol{x}) \cdot \boldsymbol{e}/\sqrt{s})(\bar{\boldsymbol{u}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{x})) \\ &\qquad - \tilde{\omega}(\boldsymbol{y}, \boldsymbol{\xi}) F_1''(\boldsymbol{r}(l, \boldsymbol{y}) \cdot \boldsymbol{e}/\sqrt{s})(\bar{\boldsymbol{u}}(\boldsymbol{y}) - \bar{\boldsymbol{v}}(\boldsymbol{y}))| dl \Big|. \end{aligned}$$

Adding and substracting $\tilde{\omega}(\boldsymbol{x}, \boldsymbol{\xi}) F_1''(\boldsymbol{r}(l, \boldsymbol{x}) \cdot \boldsymbol{e}/\sqrt{s})(\bar{\boldsymbol{u}}(\boldsymbol{y}) - \bar{\boldsymbol{v}}(\boldsymbol{y}))$ and noting $0 \le \tilde{\omega}(\boldsymbol{x}, \boldsymbol{\xi}) \le 1$ give

$$\begin{aligned} H &\le \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^\gamma} \frac{1}{\sqrt{s}} \Big| \int_0^1 |F_1''(\boldsymbol{r}(l, \boldsymbol{x}) \cdot \boldsymbol{e}/\sqrt{s})| \, |\bar{\boldsymbol{u}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{x}) - \bar{\boldsymbol{u}}(\boldsymbol{y}) + \bar{\boldsymbol{v}}(\boldsymbol{y})| \, dl \Big| \\ &\quad + \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^\gamma} \frac{1}{\sqrt{s}} \int_0^1 |(\tilde{\omega}(\boldsymbol{x}, \boldsymbol{\xi}) F_1''(\boldsymbol{r}(l, \boldsymbol{x}) \cdot \boldsymbol{e}/\sqrt{s}) - \tilde{\omega}(\boldsymbol{y}, \boldsymbol{\xi}) F_1''(\boldsymbol{r}(l, \boldsymbol{y}) \cdot \boldsymbol{e}/\sqrt{s}))| \\ &\qquad \times |\bar{\boldsymbol{u}}(\boldsymbol{y}) - \bar{\boldsymbol{v}}(\boldsymbol{y})| \, dl. \\ &=: H_1 + H_2. \end{aligned}$$

The H_1 term is estimated first. Note that $|F_1''(r)| \leq C_2$. Since $\boldsymbol{u}, \boldsymbol{v} \in C_0^{0,\gamma}(D;\mathbb{R}^d)$, it is easily seen that

$$\frac{|\bar{\boldsymbol{u}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{x}) - \bar{\boldsymbol{u}}(\boldsymbol{y}) + \bar{\boldsymbol{v}}(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma} \leq 2\|\boldsymbol{u}-\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}.$$

Therefore, we have

$$H_1 \leq \frac{2C_2}{\sqrt{s}}\|\boldsymbol{u}-\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \tag{33}$$

We now estimate H_2. We add and subtract $\tilde{\omega}(\boldsymbol{x},\boldsymbol{\xi})F_1''(\boldsymbol{r}(l,\boldsymbol{y})\cdot\boldsymbol{e}/\sqrt{s}))$ in H_2 to get

$$H_2 \leq H_3 + H_4,$$

where

$$H_3 = \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}\frac{1}{\sqrt{s}}\int_0^1 |(F_1''(\boldsymbol{r}(l,\boldsymbol{x})\cdot\boldsymbol{e}/\sqrt{s}) - F_1''(\boldsymbol{r}(l,\boldsymbol{y})\cdot\boldsymbol{e}/\sqrt{s}))|\,|\bar{\boldsymbol{u}}(\boldsymbol{y}) - \bar{\boldsymbol{v}}(\boldsymbol{y})|\,dl,$$

and

$$H_4 = \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}\frac{1}{\sqrt{s}}\int_0^1 |(\tilde{\omega}(\boldsymbol{x},\boldsymbol{\xi}) - \tilde{\omega}(\boldsymbol{y},\boldsymbol{\xi})|F_1''(\boldsymbol{r}(l,\boldsymbol{y})\cdot\boldsymbol{e}/\sqrt{s}))|\,|\bar{\boldsymbol{u}}(\boldsymbol{y}) - \bar{\boldsymbol{v}}(\boldsymbol{y})|\,dl.$$

Now we estimate H_3. Since $|F_1'''(r)| \leq C_3$ (see Eq. 22), we have

$$\begin{aligned}
&\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}|F_1''(\boldsymbol{r}(l,\boldsymbol{x})\cdot\boldsymbol{e}/\sqrt{s}) - F_1''(\boldsymbol{r}(l,\boldsymbol{y})\cdot\boldsymbol{e}/\sqrt{s})| \\
&\leq \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}\sup_r \left|F'''(r)\right| \frac{|\boldsymbol{r}(l,\boldsymbol{x})\cdot\boldsymbol{e} - \boldsymbol{r}(l,\boldsymbol{y})\cdot\boldsymbol{e}|}{\sqrt{s}} \\
&\leq \frac{C_3}{\sqrt{s}}\frac{|\boldsymbol{r}(l,\boldsymbol{x}) - \boldsymbol{r}(l,\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma} \\
&= \frac{C_3}{\sqrt{s}}\left(\frac{|1-l|\,|\bar{\boldsymbol{v}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma} + \frac{|l|\,|\bar{\boldsymbol{u}}(\boldsymbol{x}) - \bar{\boldsymbol{u}}(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}\right) \\
&\leq \frac{C_3}{\sqrt{s}}\left(\frac{|\bar{\boldsymbol{v}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma} + \frac{|\bar{\boldsymbol{u}}(\boldsymbol{x}) - \bar{\boldsymbol{u}}(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}\right).
\end{aligned} \tag{34}$$

Where we have used the fact that $|1-l| \leq 1, |l| \leq 1$, as $l \in [0,1]$. Also, note that

$$\frac{|\bar{\boldsymbol{u}}(\boldsymbol{x})-\bar{\boldsymbol{u}}(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{\gamma}} \leq 2\|\boldsymbol{u}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}$$
$$\frac{|\bar{\boldsymbol{v}}(\boldsymbol{x})-\bar{\boldsymbol{v}}(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{\gamma}} \leq 2\|\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}$$
$$|\bar{\boldsymbol{u}}(\boldsymbol{y})-\bar{\boldsymbol{v}}(\boldsymbol{y})| \leq s^{\gamma}\|\boldsymbol{u}-\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}.$$

We combine above estimates with Eq. 34 to get

$$\begin{aligned} H_3 &\leq \frac{1}{\sqrt{s}}\frac{C_3}{\sqrt{s}}\left(\|\boldsymbol{u}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} + \|\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}\right) s^{\gamma}\|\boldsymbol{u}-\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \\ &= \frac{C_3}{s^{1-\gamma}}\left(\|\boldsymbol{u}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} + \|\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}\right)\|\boldsymbol{u}-\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \end{aligned} \tag{35}$$

Next we estimate H_4. Here we add and subtract $\omega(\boldsymbol{y})\omega(\boldsymbol{x}+\epsilon\boldsymbol{\xi})$ to get

$$\begin{aligned} H_4 = \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^{\gamma}}\frac{1}{\sqrt{s}}\int_0^1 |(\omega(\boldsymbol{x},\boldsymbol{x}+\epsilon\boldsymbol{\xi})(\omega(\boldsymbol{x})-\omega(\boldsymbol{y}))+\omega(\boldsymbol{y})(\omega(\boldsymbol{x}+\epsilon\boldsymbol{\xi})-\omega(\boldsymbol{y}+\epsilon\boldsymbol{\xi})) \\ \times |F_1''(\boldsymbol{r}(l,\boldsymbol{y})\cdot\boldsymbol{e}/\sqrt{s}))|\,|\bar{\boldsymbol{u}}(\boldsymbol{y})-\bar{\boldsymbol{v}}(\boldsymbol{y})|\,dl. \end{aligned}$$

Recalling that ω belongs to $C_0^{0,\gamma}(D;\mathbb{R}^d)$ and in view of the previous estimates, a straightforward calculation gives

$$H_4 \leq \frac{4C_2}{s^{1/2-\gamma}}\|\omega\|_{C^{0,\gamma}(D;\mathbb{R}^d)}\|\boldsymbol{u}-\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \tag{36}$$

Combining Eqs. 33, 35, and 36 gives

$$\begin{aligned} H \leq \Bigg(\frac{2C_2}{\sqrt{s}} + \frac{4C_2}{s^{1/2-\gamma}}\|\omega\|_{C^{0,\gamma}(D;\mathbb{R}^d)} + \\ + \frac{C_3}{s^{1-\gamma}}\left(\|\boldsymbol{u}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} + \|\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}\right)\Bigg)\|\boldsymbol{u}-\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \end{aligned}$$

Substituting H in Eq. 28 gives

$$\begin{aligned} &\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^{\gamma}}|(-\nabla PD^{\epsilon}(\boldsymbol{u})+\nabla PD^{\epsilon}(\boldsymbol{v}))(\boldsymbol{x})-(-\nabla PD^{\epsilon}(\boldsymbol{u})+\nabla PD^{\epsilon}(\boldsymbol{v}))(\boldsymbol{y})| \\ &\leq |\frac{2}{\epsilon\omega_d}\int_{H_1(\mathbf{0})} J(|\boldsymbol{\xi}|)\frac{1}{\sqrt{s}}H\,d\boldsymbol{\xi}| \end{aligned}$$

$$\leq \left(\frac{4C_2 \bar{J}_1}{\epsilon^2} + \frac{4C_2 \bar{J}_{1-\gamma}}{\epsilon^{2-\gamma}} \|\omega\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \right.$$

$$\left. + \frac{2C_3 \bar{J}_{3/2-\gamma}}{\epsilon^{2+1/2-\gamma}} \left(\|\boldsymbol{u}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} + \|\boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \right) \right) \|\boldsymbol{u} - \boldsymbol{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \quad (37)$$

We combine Eqs. 19, 27, and 37 and get

$$\|-\nabla PD^\epsilon(\boldsymbol{u}) - (-\nabla PD^\epsilon(\boldsymbol{v}))\|_{C^{0,\gamma}}$$

$$\leq \left(\frac{4C_2 \bar{J}_1}{\epsilon^2} + \frac{2C_2 \bar{J}_{1-\gamma}}{\epsilon^{2-\gamma}} (1 + \|\omega\|_{C^{0,\gamma}}) + \frac{2C_3 \bar{J}_{3/2-\gamma}}{\epsilon^{2+1/2-\gamma}} (\|\boldsymbol{u}\|_{C^{0,\gamma}} + \|\boldsymbol{v}\|_{C^{0,\gamma}}) \right) \|\boldsymbol{u} - \boldsymbol{v}\|_{C^{0,\gamma}}$$

$$\leq \frac{\bar{C}_1 + \bar{C}_2 \|\omega\|_{C^{0,\gamma}} + \bar{C}_3 (\|\boldsymbol{u}\|_{C^{0,\gamma}} + \|\boldsymbol{v}\|_{C^{0,\gamma}})}{\epsilon^{2+\alpha(\gamma)}} \|\boldsymbol{u} - \boldsymbol{v}\|_{C^{0,\gamma}} \quad (38)$$

where we introduce new constants $\bar{C}_1, \bar{C}_2, \bar{C}_3$. We let $\alpha(\gamma) = 0$, if $\gamma \geq 1/2$, and $\alpha(\gamma) = 1/2 - \gamma$, if $\gamma \leq 1/2$. One can easily verify that, for all $\gamma \in (0, 1]$ and $0 < \epsilon \leq 1$,

$$\max \left\{ \frac{1}{\epsilon^2}, \frac{1}{\epsilon^{2+1/2-\gamma}}, \frac{1}{\epsilon^{2-\gamma}} \right\} \leq \frac{1}{\epsilon^{2+\alpha(\gamma)}}$$

To complete the proof, we combine Eqs. 38 and 18 and get

$$\|F^\epsilon(y, t) - F^\epsilon(z, t)\|_X \leq \frac{L_1 + L_2(\|\omega\|_{C^{0,\gamma}} + \|y\|_X + \|z\|_X)}{\epsilon^{2+\alpha(\gamma)}} \|y - z\|_X.$$

This proves the Lipschitz continuity of $F^\epsilon(y, t)$ on any bounded subset of X. The bound on $F^\epsilon(y, t)$ (see Eq. 17) follows easily from Eq. 25. This completes the proof of Proposition 1.

Existence of Solution in Hölder Space

In this section, we prove Theorem 1. We begin by proving a local existence theorem. We then show that the local solution can be continued uniquely in time to recover Theorem 1.

The existence and uniqueness of local solutions is stated in the following theorem.

Theorem 2 (Local existence and uniqueness). *Given* $X = C_0^{0,\gamma}(D; \mathbb{R}^d) \times C_0^{0,\gamma}(D; \mathbb{R}^d)$, $\boldsymbol{b}(t) \in C_0^{0,\gamma}(D; \mathbb{R}^d)$, *and initial data* $x_0 = (\boldsymbol{u}_0, \boldsymbol{v}_0) \in X$. *We suppose that* $\boldsymbol{b}(t)$ *is continuous in time over some time interval* $I_0 = (-T, T)$

and satisfies $\sup_{t\in I_0}\|\boldsymbol{b}(t)\|_{C^{0,\gamma}(D;\mathbb{R}^d)} < \infty$. *Then, there exists a time interval* $I' = (-T', T') \subset I_0$ *and unique solution* $y = (y^1, y^2)$ *such that* $y \in C^1(I'; X)$ *and*

$$y(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau)\, d\tau, \ \textit{for}\ t \in I' \tag{39}$$

or equivalently

$$y'(t) = F^\epsilon(y(t), t), \textit{with}\ y(0) = x_0,\ \textit{for}\ t \in I'$$

where $y(t)$ *and* $y'(t)$ *are Lipschitz continuous in time for* $t \in I' \subset I_0$.

To prove Theorem 2, we proceed as follows. We write $y(t) = (y^1(t), y^2(t))$ and $||y||_X = ||y^1(t)||_{C^{0,\gamma}} + ||y^2(t)||_{C^{0,\gamma}}$. Define the ball $B(0, R)=\{y \in X : ||y||_X<R\}$ and choose $R > ||x_0||_X$. Let $r = R - \|x_0\|_X$ and we consider the ball $B(x_0, r)$ defined by

$$B(x_0, r) = \{y \in X : ||y - x_0||_X < r\} \subset B(0, R), \tag{40}$$

(see Fig. 6).

To recover the existence and uniqueness, we introduce the transformation

$$S_{x_0}(y)(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau)\, d\tau.$$

Introduce $0 < T' < T$ and the associated set $Y(T')$ of Hölder continuous functions taking values in $B(x_0, r)$ for $I' = (-T', T') \subset I_0 = (-T, T)$. The goal is to find appropriate interval $I' = (-T', T')$ for which S_{x_0} maps into the corresponding set $Y(T')$. Writing out the transformation with $y(t) \in Y(T')$ gives

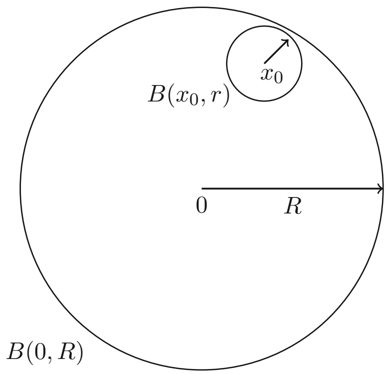

Fig. 6 Geometry

$$S^1_{x_0}(y)(t) = x^1_0 + \int_0^t y^2(\tau)\,d\tau \tag{41}$$

$$S^2_{x_0}(y)(t) = x^2_0 + \int_0^t (-\nabla PD^\epsilon(y^1(\tau)) + \boldsymbol{b}(\tau))\,d\tau, \tag{42}$$

and there is a positive constant $K = C/\epsilon^{2+\alpha(\gamma)}$ (see Eq. 17) independent of $y^1(t)$, for $-T' < t < T'$, such that estimation in Eq. 42 gives

$$||S^2_{x_0}(y)(t) - x^2_0||_{C^{0,\gamma}} \leq (K(1 + \frac{1}{\epsilon^\gamma} + \sup_{t\in(-T',T')} ||y^1(t)||_{C^{0,\gamma}}) + \sup_{t\in(-T,T)} ||\boldsymbol{b}(t)||_{C^{0,\gamma}})T' \tag{43}$$

and from Eq. 41

$$||S^1_{x_0}(y)(t) - x^1_0||_{C^{0,\gamma}} \leq \sup_{t\in(-T',T')} ||y^2(t)||_{C^{0,\gamma}} T'. \tag{44}$$

We write $b = \sup_{t\in I_0} ||\boldsymbol{b}(t)||_{C^{0,\gamma}}$ and adding Eqs. 43 and 44 gives the upper bound

$$||S_{x_0}(y)(t) - x_0||_X \leq (K(1 + \frac{1}{\epsilon^\gamma} + \sup_{t\in(-T',T')} ||y(t)||_X) + b)T'. \tag{45}$$

Since $B(x_0, r) \subset B(0, R)$ (see Eq. 40), we make the choice T' so that

$$||S_{x_0}(y)(t) - x_0||_X \leq ((K(1 + \frac{1}{\epsilon^\gamma} + R) + b)T' < r = R - ||x_0||_X. \tag{46}$$

For this choice we see that

$$T' < \theta(R) = \frac{R - ||x_0||_X}{K(R + 1 + \frac{1}{\epsilon^\gamma}) + b}. \tag{47}$$

Now it is easily seen that $\theta(R)$ is increasing with $R > 0$ and

$$\lim_{R\to\infty} \theta(R) = \frac{1}{K}. \tag{48}$$

So given R and $||x_0||_X$, we choose T' according to

$$\frac{\theta(R)}{2} < T' < \theta(R), \tag{49}$$

and set $I' = (-T', T')$. We have found the appropriate time domain I' such that the transformation $S_{x_0}(y)(t)$ as defined in Eqs. 41 and 42 maps $Y(T')$ into itself. We now proceed using standard arguments (see, e.g., Driver 2003, Theorem 6.10) to complete the proof of existence and uniqueness of solution for given initial data x_0 over the interval $I' = (-T', T')$.

We now prove Theorem 1. From the proof of Theorem 2 above, we see that a unique local solution exists over a time domain $(-T', T')$ with $\frac{\theta(R)}{2} < T'$. Since $\theta(R) \nearrow 1/K$ as $R \nearrow \infty$, we can fix a tolerance $\eta > 0$ so that $[(1/2K) - \eta] > 0$. Then given any initial condition with bounded Hölder norm and $b = \sup_{t\in[-T,T)} ||\boldsymbol{b}(t)||_{C^{0,\gamma}}$, we can choose R sufficiently large so that $||x_0||_X < R$ and $0 < (1/2K)) - \eta < T'$. Thus we can always find local solutions for time intervals $(-T', T')$ for T' larger than $[(1/2K) - \eta] > 0$. Therefore we apply the local existence and uniqueness result to uniquely continue local solutions up to an arbitrary time interval $(-T, T)$.

Existence of Solutions in the Sobolev Space H^2

We start by recalling that the space $H_0^2(D; \mathbb{R}^d)$ is the closure in the H^2 norm of twice differentiable functions with compact support in D. We denote the norm in H^m by $|| \cdot ||_m$, $m = 1, 2$, and the L^∞ norm by $|| \cdot ||_\infty$. In this section, we find that solutions of peridynamic evolutions exist for almost all times in $H_0^2(D; \mathbb{R}^d) \cap L^\infty(D; \mathbb{R}^d)$. For the sake of convenience, we let W denote the $H_0^2(D; \mathbb{R}^d) \cap L^\infty(D; \mathbb{R}^d)$ space. The norm on W is defined as

$$||u||_W := ||u||_2 + ||u||_\infty. \tag{50}$$

We will assume that $u \in H_0^2(D; \mathbb{R}^d)$ is extended by zero outside D; therefore, $u = 0, \nabla u = 0, \nabla^2 u = 0$ for $x \notin D$ and $||u||_{H^2(D;\mathbb{R}^d)} = ||u||_{H^2(\mathbb{R}^d;\mathbb{R}^d)}$.

Noting the Sobolev embedding property of $u \in H_0^2(D; \mathbb{R}^d)$ (see Theorem 2.31, Demengel and Demengel 2012) given by

$$||\nabla u||_{L^q(D;\mathbb{R}^{d\times d})} \leq C_e ||u||_{H_0^2(D;\mathbb{R}^d)} \tag{51}$$

for any q such that $2 \leq q \leq 6$ in case of $d = 3$ and $2 \leq q < \infty$ in case of $d = 2$. Constant C_e is independent of u.

In what follows, we will first prove the Lipschitz bound on $-\nabla PD^\epsilon(u)$, and then using Lipschitz bound, we will show the local existence of solution u in W. We write the peridynamic equation as an equivalent first-order system with $y_1(t) = u(t)$ and $y_2(t) = v(t)$ with $v(t) = \dot{u}(t)$. Let $y = (y_1, y_2)^T$ where $y_1, y_2 \in W$ and let $F^\epsilon(y, t) = (F_1^\epsilon(y, t), F_2^\epsilon(y, t))^T$ such that

$$F_1^\epsilon(y, t) := y_2, \tag{52}$$

$$F_2^\epsilon(y, t) := -\nabla PD^\epsilon(y_1) + b(t). \tag{53}$$

The initial boundary value is equivalent to the initial boundary value problem for the first-order system given by

$$\dot{y}(t) = F^{\epsilon}(y,t), \tag{54}$$

with initial condition given by $y(0) = (u_0, v_0)^T \in W \times W$.

Theorem 3 (Lipschitz bound on peridynamic force). *For any $u, v \in W$, we have*

$$\begin{aligned} &|| -\nabla PD^{\epsilon}(u) - (-\nabla PD^{\epsilon}(v))||_W \\ &\leq \frac{\bar{L}_1 + \bar{L}_2(||u||_W + ||v||_W) + \bar{L}_3(||u||_W + ||v||_W)^2}{\epsilon^3}||u-v||_W \end{aligned} \tag{55}$$

where constants $\bar{L}_1, \bar{L}_2, \bar{L}_3$ are independent of ϵ, u, and v and are defined in (96). Also, for $u \in W$, we have

$$|| -\nabla PD^{\epsilon}(u)||_W \leq \frac{\bar{L}_4||u||_W + \bar{L}_5||u||_W^2}{\epsilon^{5/2}}, \tag{56}$$

where constants are independent of ϵ and u and are defined in (105).

We state the theorem which shows the existence and uniqueness of solution in any given finite time interval $I_0 = (-T, T)$.

Theorem 4 (Existence and uniqueness of solutions over finite time intervals). *For any initial condition $x_0 \in X = W \times W$, time interval $I_0 = (-T, T)$, and right-hand side $b(t)$ continuous in time for $t \in I_0$ such that $b(t)$ satisfies $\sup_{t \in I_0} ||b(t)||_W < \infty$, there is a unique solution $y(t) \in C^1(I_0; X)$ of*

$$y(t) = x_0 + \int_0^t F^{\epsilon}(y(\tau), \tau)\, d\tau,$$

or equivalently

$$y'(t) = F^{\epsilon}(y(t), t), \text{with } y(0) = x_0,$$

where $y(t)$ and $y'(t)$ are Lipschitz continuous in time for $t \in I_0$.

The proof of the Lipschitz continuity and existence is established in the following section.

Lipschitz Continuity in the H^2 Norm and Existence of an H^2 Solution

In this section we prove Theorems 3 and 4. To simplify the presentation, we denote the peridynamic force $-\nabla PD^{\epsilon}(u)$ by simply $P(u)$. Recall that we denote $H_0^2(D;\mathbb{R}^d) \cap L^{\infty}(D;\mathbb{R}^d)$ by W and

$$||u||_W = ||u|| + ||\nabla u|| + ||\nabla^2 u|| + ||u||_\infty.$$

We need to analyze $||P(u) - P(v)||_W$.

We use the following short notations:

$$s_\xi = \epsilon|\xi|,\ e_\xi = \frac{\xi}{|\xi|},\ \bar{J}_\alpha = \frac{1}{\omega_d}\int_{H_1(0)} J(|\xi|)\frac{1}{|\xi|^\alpha}d\xi,$$
$$S_\xi(u) = \frac{u(x+\epsilon\xi) - u(x)}{s_\xi}\cdot e_\xi,$$
$$S_\xi(\nabla u) = \nabla S_\xi(u) = \frac{\nabla u^T(x+\epsilon\xi) - \nabla u^T(x)}{s_\xi}e_\xi,$$
$$S_\xi(\nabla^2 u) = \nabla S_\xi(\nabla u) = \nabla\left[\frac{\nabla u^T(x+\epsilon\xi) - \nabla u^T(x)}{s_\xi}e_\xi\right].$$

In indicial notation, we have

$$S_\xi(\nabla u)_i = \frac{u_{k,i}(x+\epsilon\xi) - u_{k,i}(x)}{s_\xi}(e_\xi)_k,$$
$$S_\xi(\nabla^2 u)_{ij} = \left[\frac{u_{k,i}(x+\epsilon\xi) - u_{k,i}(x)}{s_\xi}(e_\xi)_k\right]_{,j} = \frac{u_{k,ij}(x+\epsilon\xi) - u_{k,ij}(x)}{s_\xi}(e_\xi)_k \tag{57}$$

and

$$[e_\xi \otimes S_\xi(\nabla^2 u)]_{ijk} = (e_\xi)_i S_\xi(\nabla^2 u)_{jk}, \tag{58}$$

where $u_{i,j} = (\nabla u)_{ij}$, $u_{k,ij} = (\nabla^2 u)_{kij}$, and $(e_\xi)_k = \xi_k/|\xi|$.

Peridynamic Force

Let $F_1(r) := f(r^2)$ where f is described in section "Problem Formulation with Bond-Based Nonlinear Potentials". We have $F_1'(r) = f'(r^2)2r$. Thus, $2Sf'(\epsilon|\xi|S^2) = F_1'(\sqrt{\epsilon|\xi|}S)/\sqrt{\epsilon|\xi|}$. We define the following constants related to nonlinear potential

$$C_1 := \sup_r |F_1'(r)|,\ C_2 := \sup_r |F_1''(r)|,\ C_3 := \sup_r |F_1'''(r)|,\ C_4 := \sup_r |F_1''''(r)|.$$

The potential function f as chosen here satisfies $C_1, C_2, C_3, C_4 < \infty$. Let

$$\bar{\omega}_\xi(x) = \omega(x)\omega(x+\epsilon\xi), \tag{59}$$

and we choose ω such that

$$|\nabla \bar{\omega}_\xi| \leq C_{\omega_1} < \infty \quad \text{and} \quad |\nabla^2 \bar{\omega}_\xi| \leq C_{\omega_2} < \infty. \tag{60}$$

With notations described so far, we write peridynamic force $P(u)$ as

$$P(u)(x) = \frac{2}{\epsilon \omega_d} \int_{H_1(0)} \bar{\omega}_\xi(x) J(|\xi|) \frac{F_1'(\sqrt{s_\xi} S_\xi(u))}{\sqrt{s_\xi}} e_\xi d\xi. \tag{61}$$

The gradient of $P(u)(x)$ is given by

$$\begin{aligned}\nabla P(u)(x) &= \frac{2}{\epsilon \omega_d} \int_{H_1(0)} \bar{\omega}_\xi(x) J(|\xi|) F_1''(\sqrt{s_\xi} S_\xi(u)) e_\xi \otimes \nabla S_\xi(u) d\xi \\ &+ \frac{2}{\epsilon \omega_d} \int_{H_1(0)} J(|\xi|) \frac{F_1'(\sqrt{s_\xi} S_\xi(u))}{\sqrt{s_\xi}} e_\xi \otimes \nabla \bar{\omega}_\xi(x) d\xi \\ &= g_1(u)(x) + g_2(u)(x), \end{aligned} \tag{62}$$

where we denote first and second term as $g_1(u)(x)$ and $g_2(u)(x)$, respectively. We also have

$$\begin{aligned}\nabla^2 P(u)(x) &= \frac{2}{\epsilon \omega_d} \int_{H_1(0)} \bar{\omega}_\xi(x) J(|\xi|) F_1''(\sqrt{s_\xi} S_\xi(u)) e_\xi \otimes S_\xi(\nabla^2 u) d\xi \\ &+ \frac{2}{\epsilon \omega_d} \int_{H_1(0)} \bar{\omega}_\xi(x) J(|\xi|) \sqrt{s_\xi} F_1'''(\sqrt{s_\xi} S_\xi(u)) e_\xi \otimes S_\xi(\nabla u) \otimes S_\xi(\nabla u) d\xi \\ &+ \frac{2}{\epsilon \omega_d} \int_{H_1(0)} J(|\xi|) F_1''(\sqrt{s_\xi} S_\xi(u)) e_\xi \otimes S_\xi(\nabla u) \otimes \nabla \bar{\omega}_\xi(x) d\xi \\ &+ \frac{2}{\epsilon \omega_d} \int_{H_1(0)} J(|\xi|) \frac{F_1'(\sqrt{s_\xi} S_\xi(u))}{\sqrt{s_\xi}} e_\xi \otimes \nabla^2 \bar{\omega}_\xi(x) d\xi \\ &+ \frac{2}{\epsilon \omega_d} \int_{H_1(0)} J(|\xi|) F_1''(\sqrt{s_\xi} S_\xi(u)) e_\xi \otimes \nabla \bar{\omega}_\xi(x) \otimes S_\xi(\nabla u) d\xi \\ &= h_1(u)(x) + h_2(u)(x) + h_3(u)(x) + h_4(u)(x) + h_5(u)(x) \end{aligned} \tag{63}$$

where we denote first, second, third, fourth, and fifth terms as h_1, h_2, h_2, h_4, and h_5, respectively. Estimating $||P(u) - P(v)||$ and $||P(u) - P(v)||_\infty$. From (61), we have

$$
\begin{aligned}
&|P(u)(x) - P(v)(x)| \\
&\quad \le \frac{2}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)\frac{1}{\sqrt{s_\xi}}|F_1'(\sqrt{s_\xi}S_\xi(u)) - F_1'(\sqrt{s_\xi}S_\xi(v))|d\xi \\
&\quad \le \frac{2}{\epsilon\omega_d}\left(\sup_r |F_1'(r)|\right)\int_{H_1(0)} J(|\xi|)\frac{1}{\sqrt{s_\xi}}|\sqrt{s_\xi}S_\xi(u) - \sqrt{s_\xi}S_\xi(v)|d\xi \\
&\quad = \frac{2C_2}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)|S_\xi(u) - S_\xi(v)|d\xi, \qquad (64)
\end{aligned}
$$

where we used the fact that $|\bar{\omega}_\xi(x)| \le 1$ and $|F_1'(r_1) - F_1'(r_2)| \le C_2|r_1 - r_2|$. Since

$$
|S_\xi(u) - S_\xi(v)| \le \frac{|u(x+\epsilon\xi) - v(x+\epsilon\xi)| + |u(x) - v(x)|}{\epsilon|\xi|}
$$

we have

$$
||P(u) - P(v)||_\infty \le \frac{2C_2}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)\frac{2||u-v||_\infty}{\epsilon|\xi|}d\xi = \frac{L_1}{\epsilon^2}||u-v||_W \qquad (65)
$$

where we let $L_1 := 4C_2\bar{J}_1$.

From (64), we have

$$
\begin{aligned}
&||P(u) - P(v)||^2 \\
&\le \int_D \left(\frac{2C_2}{\epsilon\omega_d}\right)^2 \int_{H_1(0)}\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|}\frac{J(|\eta|)}{|\eta|}|\xi||S_\xi(u) - S_\xi(v)||\eta||S_\eta(u) - S_\eta(v)|d\xi d\eta dx.
\end{aligned}
$$

Using the identities $|a||b| \le |a|^2/2 + |b|^2/2$ and $(a+b)^2 \le 2a^2 + 2b^2$, we get

$$
\begin{aligned}
&||P(u) - P(v)||^2 \\
&\le \int_D \left(\frac{2C_2}{\epsilon\omega_d}\right)^2 \int_{H_1(0)}\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|}\frac{J(|\eta|)}{|\eta|}\frac{|\xi|^2|S_\xi(u) - S_\xi(v)|^2 + |\eta|^2|S_\eta(u) - S_\eta(v)|^2}{2}d\xi d\eta dx \\
&= 2\int_D \left(\frac{2C_2}{\epsilon\omega_d}\right)^2 \int_{H_1(0)}\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|}\frac{J(|\eta|)}{|\eta|}\frac{|\xi|^2|S_\xi(u) - S_\xi(v)|^2}{2}d\xi d\eta dx \\
&= \int_D \left(\frac{2C_2}{\epsilon\omega_d}\right)^2 \omega_d\bar{J}_1\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|}|\xi|^2\frac{2|u(x+\epsilon\xi) - v(x+\epsilon\xi)|^2 + 2|u(x) - v(x)|^2}{\epsilon^2|\xi|^2}d\xi dx \\
&= \left(\frac{2C_2}{\epsilon\omega_d}\right)^2 \omega_d\bar{J}_1\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|}\frac{1}{\epsilon^2}\left[2\int_D \left(|u(x+\epsilon\xi) - v(x+\epsilon\xi)|^2 + |u(x) - v(x)|^2\right)dx\right]d\xi \\
&\le \left(\frac{2C_2}{\epsilon\omega_d}\right)^2 \omega_d\bar{J}_1\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|}\frac{1}{\epsilon^2}\left[4||u-v||^2\right]d\xi, \qquad (66)
\end{aligned}
$$

where we used symmetry wrt ξ and η in second equation. This gives

$$||P(u) - P(v)|| \leq \frac{L_1}{\epsilon^2}||u - v|| \leq \frac{L_1}{\epsilon^2}||u - v||_W. \tag{67}$$

Estimating $||\nabla P(u) - \nabla P(v)||$. From (62), we have

$$||\nabla P(u) - \nabla P(v)|| \leq ||g_1(u) - g_1(v)|| + ||g_2(u) - g_2(v)||.$$

Using $|\bar{\omega}_\xi(x)| \leq 1$, we get

$$\begin{aligned}
&|g_1(u)(x) - g_1(v)(x)| \\
&\leq \frac{2}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)|F_1''(\sqrt{s_\xi}S_\xi(u))\nabla S_\xi(u) - F_1''(\sqrt{s_\xi}S_\xi(v))\nabla S_\xi(v)|d\xi \\
&\leq \frac{2}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)|F_1''(\sqrt{s_\xi}S_\xi(u)) - F_1''(\sqrt{s_\xi}S_\xi(v))||\nabla S_\xi(u)|d\xi \\
&+ \frac{2}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)|F_1''(\sqrt{s_\xi}S_\xi(v))||\nabla S_\xi(u) - \nabla S_\xi(v)|d\xi \\
&\leq \frac{2C_3}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)\sqrt{s_\xi}|S_\xi(u) - S_\xi(v)||\nabla S_\xi(u)|d\xi \\
&+ \frac{2C_2}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)|\nabla S_\xi(u) - \nabla S_\xi(v)|d\xi \\
&= I_1(x) + I_2(x)
\end{aligned} \tag{68}$$

where we denote first and second term as $I_1(x)$ and $I_2(x)$. Proceeding similar to (66), we can show

$$\begin{aligned}
||I_1||^2 &= \int_D \left(\frac{2C_3}{\epsilon\omega_d}\right)^2 \int_{H_1(0)}\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}}\frac{J(|\eta|)}{|\eta|^{3/2}}|\xi|^{3/2}|\eta|^{3/2}\sqrt{s_\xi}\sqrt{s_\eta} \\
&\quad \times |S_\xi(u) - S_\xi(v)||\nabla S_\xi(u)||S_\eta(u) - S_\eta(v)||\nabla S_\eta(u)|d\xi d\eta dx \\
&\leq \int_D \left(\frac{2C_3}{\epsilon\omega_d}\right)^2 \omega_d \bar{J}_{3/2}\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}}|\xi|^3 s_\xi|S_\xi(u) - S_\xi(v)|^2|\nabla S_\xi(u)|^2 d\xi dx.
\end{aligned} \tag{69}$$

Now

$$\int_D |S_\xi(u) - S_\xi(v)|^2|\nabla S_\xi(u)|^2 dx$$

$$\leq \frac{4||u-v||_\infty^2}{\epsilon^2|\xi|^2}\frac{1}{\epsilon^2|\xi|^2}\int_D 2(|\nabla u(x+\epsilon\xi)|^2 + |\nabla u(x)|^2)dx$$
$$\leq \frac{16||\nabla u||^2||u-v||_\infty^2}{\epsilon^4|\xi|^4} \leq \frac{16||u||_W^2}{\epsilon^4|\xi|^4}||u-v||_W^2.$$

Substituting above in (69) to get

$$||I_1||^2 \leq \left(\frac{2C_3}{\epsilon\omega_d}\right)^2 \omega_d \bar{J}_{3/2}\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}}|\xi|^3\epsilon|\xi|\frac{16||u||_W^2}{\epsilon^4|\xi|^4}||u-v||_W^2 d\xi$$
$$= \left(\frac{8C_3\bar{J}_{3/2}||u||_W}{\epsilon^{5/2}}\right)^2 ||u-v||_W^2.$$

Let $L_2 = 8C_3\bar{J}_{3/2}$ to write

$$||I_1|| \leq \frac{L_2(||u||_W + ||v||_W)}{\epsilon^{5/2}}||u-v||_W. \tag{70}$$

Similarly

$$||I_2||^2 = \int_D \left(\frac{2C_2}{\epsilon\omega_d}\right)^2 \int_{H_1(0)}\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|}\frac{J(|\eta|)}{|\eta|}|\xi||\eta|$$
$$\times |\nabla S_\xi(u) - \nabla S_\xi(v)||\nabla S_\eta(u) - \nabla S_\eta(v)| d\xi d\eta dx$$
$$\leq \left(\frac{2C_2}{\epsilon\omega_d}\right)^2 \omega_d \bar{J}_1 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|}|\xi|^2\left[\int_D |\nabla S_\xi(u) - \nabla S_\xi(v)|^2 dx\right] d\xi.$$

This gives

$$||I_2|| \leq \frac{4C_2\bar{J}_1}{\epsilon^2}||u-v||_W = \frac{L_1}{\epsilon^2}||u-v||_W. \tag{71}$$

Thus

$$||g_1(u) - g_1(v)|| \leq \frac{\sqrt{\epsilon}L_1 + L_2(||u||_W + ||v||_W)}{\epsilon^{5/2}}||u-v||_W. \tag{72}$$

We now work on $|g_2(u)(x) - g_2(v)(x)|$ (see (62)). Noting the bound on $\nabla\bar{\omega}_\xi$, we get

$$|g_2(u)(x) - g_2(v)(x)|$$

$$
\begin{aligned}
&= \left| \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \left[\frac{F_1'(\sqrt{s_\xi} S_\xi(u))}{\sqrt{s_\xi}} - \frac{F_1'(\sqrt{s_\xi} S_\xi(v))}{\sqrt{s_\xi}} \right] e_\xi \otimes \nabla \bar{\omega}_\xi(s) d\xi \right| \\
&\leq \frac{2C_{\omega_1}}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \left| \frac{F_1' \sqrt{s_\xi} S_\xi(u))}{\sqrt{s_\xi}} - \frac{F_1'(\sqrt{s_\xi} S_\xi(v))}{\sqrt{s_\xi}} \right| d\xi \\
&\leq \frac{2C_{\omega_1} C_2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |S_\xi(u) - S_\xi(v)| d\xi. \qquad (73)
\end{aligned}
$$

Above is similar to (64) and therefore we get

$$
||g_2(u) - g_2(v)|| \leq \frac{4C_{\omega_1} C_2 \bar{J}_1}{\epsilon^2} ||u - v||_W = \frac{C_{\omega_1} L_1}{\epsilon^2} ||u - v||_W. \qquad (74)
$$

Combining (72) and (74) to write

$$
||\nabla P(u) - \nabla P(v)|| \leq \frac{\sqrt{\epsilon}(1 + C_{\omega_1}) L_1 + L_2(||u||_W + ||v||_W)}{\epsilon^{5/2}} ||u - v||_W. \qquad (75)
$$

Estimating $||\nabla^2 P(u) - \nabla^2 P(v)||$. From (63), we have

$$
\begin{aligned}
&||\nabla^2 P(u) - \nabla^2 P(v)|| \\
&\leq ||h_1(u) - h_1(v)|| + ||h_2(u) - h_2(v)|| + ||h_3(u) - h_3(v)|| \\
&+ ||h_4(u) - h_4(v)|| + ||h_5(u) - h_5(v)||. \qquad (76)
\end{aligned}
$$

We can show, using the fact $|\bar{\omega}_\xi(x)| \leq 1$ and $|F_1''(r_1) - F_1''(r_2)| \leq C_3 |r_1 - r_2|$, that

$$
\begin{aligned}
|h_1(u)(x) - h_1(v)(x)| &\leq \frac{2C_3}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \sqrt{s_\xi} |S_\xi(u) - S_\xi(v)| |S_\xi(\nabla^2 u)| d\xi \\
&+ \frac{2C_2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |S_\xi(\nabla^2 u) - S_\xi(\nabla^2 v)| d\xi \\
&= I_3(x) + I_4(x). \qquad (77)
\end{aligned}
$$

Following similar steps used above, we can show

$$
||I_3|| \leq \frac{8C_3 \bar{J}_{3/2} ||u||_W}{\epsilon^{5/2}} ||u - v||_W \leq \frac{L_2(||u||_W + ||v||_W)}{\epsilon^{5/2}} ||u - v||_W \qquad (78)
$$

and

$$
||I_4|| \leq \frac{4C_2 \bar{J}_1}{\epsilon^2} ||u - v||_W = \frac{L_1}{\epsilon^2} ||u - v||_W, \qquad (79)
$$

where $L_1 = 4C_2\bar{J}_1$, $L_2 = 8C_3\bar{J}_{3/2}$.

Next we focus on $|h_2(u)(x) - h_2(v)(x)|$ and get

$$
\begin{aligned}
&|h_2(u)(x) - h_2(v)(x)| \\
&\quad \le \frac{2}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)\sqrt{s_\xi}|F_1'''(\sqrt{s_\xi}S_\xi(u)) - F_1'''(\sqrt{s_\xi}S_\xi(v))||S_\xi(\nabla u)|^2 d\xi \\
&\quad + \frac{2}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)\sqrt{s_\xi}|F_1'''(\sqrt{s_\xi}S_\xi(v))||S_\xi(\nabla u)\otimes S_\xi(\nabla u) \\
&\quad - S_\xi(\nabla v)\otimes S_\xi(\nabla v)|d\xi \le \frac{2C_4}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)s_\xi|S_\xi(u) - S_\xi(v)||S_\xi(\nabla u)|^2 d\xi \\
&\quad + \frac{2C_3}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)\sqrt{s_\xi}|S_\xi(\nabla u)\otimes S_\xi(\nabla u) - S_\xi(\nabla v)\otimes S_\xi(\nabla v)|d\xi \\
&\quad = I_5(x) + I_6(x).
\end{aligned}
\tag{80}
$$

Proceeding as below for $||I_5||^2$

$$
\begin{aligned}
&||I_5||^2 \\
&\le \int_D \left(\frac{2C_4}{\epsilon\omega_d}\right)^2 \int_{H_1(0)}\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^2}\frac{J(|\eta|)}{|\eta|^2}|\xi|^2 s_\xi |\eta|^2 s_\eta \\
&\times |S_\xi(u) - S_\xi(v)||S_\xi(\nabla u)|^2|S_\eta(u) - S_\eta(v)||S_\eta(\nabla u)|^2 d\xi d\eta dx \\
&\le \int_D \left(\frac{2C_4}{\epsilon\omega_d}\right)^2 \omega_d \bar{J}_2 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^2}|\xi|^4 s_\xi^2 |S_\xi(u) - S_\xi(v)|^2|S_\xi(\nabla u)|^4 d\xi dx \\
&\le \left(\frac{2C_4}{\epsilon\omega_d}\right)^2 \omega_d \bar{J}_2 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^2}|\xi|^4 s_\xi^2 \frac{4||u-v||_\infty^2}{\epsilon^2|\xi|^2}\left[\int_D |S_\xi(\nabla u)|^4 dx\right] d\xi.
\end{aligned}
\tag{81}
$$

We estimate the term in square bracket. Using the identity $(|a|+|b|)^4 \le (2|a|^2 + 2|b|^2)^2 \le 8|a|^4 + 8|b|^4$, we have

$$
\begin{aligned}
\int_D |S_\xi(\nabla u)|^4 dx &\le \frac{8}{\epsilon^4|\xi|^4}\int_D (|\nabla u(x+\epsilon\xi)|^4 + |\nabla u(x)|^4)dx \\
&\le \frac{16}{\epsilon^4|\xi|^4}||\nabla u||^4_{L^4(D;\mathbb{R}^{d\times d})}.
\end{aligned}
\tag{82}
$$

where $||u||_{L^4(D,\mathbb{R}^d)} = \left[\int_D |u|^4 dx\right]^{1/4}$. Using Sobolev embedding property of $u \in H_0^2(D;\mathbb{R}^d)$ as mentioned in (51), we get

$$\int_D |S_\xi(\nabla u)|^4 dx \leq \frac{16}{\epsilon^4|\xi|^4} C_e^4 ||\nabla u||^4_{H^1(D;\mathbb{R}^{d\times d})} \leq \frac{16C_e^4}{\epsilon^4|\xi|^4}||u||^4_W. \tag{83}$$

Substituting above to get

$$||I_5||^2 \leq \left(\frac{2C_4}{\epsilon\omega_d}\right)^2 \omega_d \bar{J}_2 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^2}|\xi|^4 s_\xi^2 \frac{4||u-v||^2_\infty}{\epsilon^2|\xi|^2}\frac{16C_e^4}{\epsilon^4|\xi|^4}||u||^4_W d\xi$$

Above gives

$$||I_5|| \leq \frac{16C_4C_e^2\bar{J}_2||u||^2_W}{\epsilon^3}||u-v||_W \leq \frac{L_3(||u||_W+||v||_W)^2}{\epsilon^3}||u-v||_W \tag{84}$$

where we let $L_3 = 16C_4C_e^2\bar{J}_2$.

Next, using

$$|S_\xi(\nabla u)\otimes S_\xi(\nabla u) - S_\xi(\nabla v)\otimes S_\xi(\nabla v)| \leq (|S_\xi(\nabla u)|+|S_\xi(\nabla v)|)|S_\xi(\nabla u)-S_\xi(\nabla v)|$$

to estimate $||I_6||$ as follows:

$$\begin{aligned}
&||I_6||^2\\
&\leq \int_D \left(\frac{2C_3}{\epsilon\omega_d}\right)^2 \int_{H_1(0)}\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}}\frac{J(|\eta|)}{|\eta|^{3/2}}|\xi|^{3/2}|\eta|^{3/2}\sqrt{s_\xi s_\eta}\\
&\times(|S_\xi(\nabla u)|+|S_\xi(\nabla v)|)|S_\xi(\nabla u)-S_\xi(\nabla v)|\times(|S_\eta(\nabla u)|+|S_\eta(\nabla v)|)|S_\eta(\nabla u)\\
&-S_\eta(\nabla v)|d\xi d\eta dx \leq \int_D\left(\frac{2C_3}{\epsilon\omega_d}\right)^2\omega_d\bar{J}_{3/2}\int_{H_1(0)}\frac{J(|\xi|)}{|\xi|^{3/2}}|\xi|^3\epsilon|\xi|(|S_\xi(\nabla u)|\\
&+|S_\xi(\nabla v)|)^2|S_\xi(\nabla u)-S_\xi(\nabla v)|^2 d\xi dx = \left(\frac{2C_3}{\epsilon\omega_d}\right)^2\omega_d\bar{J}_{3/2}\int_{H_1(0)}\frac{J(|\xi|)}{|\xi|^{3/2}}|\xi|^3\epsilon|\xi|\\
&\left[\int_D(|S_\xi(\nabla u)|+|S_\xi(\nabla v)|)^2|S_\xi(\nabla u)-S_\xi(\nabla v)|^2dx\right]d\xi.
\end{aligned}\tag{85}$$

We focus on the term in square bracket. Using Holder inequality, we have

$$\begin{aligned}
&\int_D(|S_\xi(\nabla u)|+|S_\xi(\nabla v)|)^2|S_\xi(\nabla u)-S_\xi(\nabla v)|^2dx\\
&\leq\left(\int_D(|S_\xi(\nabla u)|+|S_\xi(\nabla v)|)^4dx\right)^{1/2}\left(\int_D|S_\xi(\nabla u)-S_\xi(\nabla v)|^4dx\right)^{1/2}.
\end{aligned}\tag{86}$$

Using $(|a| + |b|)^4 \leq 8|a|^4 + 8|b|^4$, we get

$$\int_D (|S_\xi(\nabla u)| + |S_\xi(\nabla v)|)^4 dx \leq 8\left[\int_D |S_\xi(\nabla u)|^4 dx + \int_D |S_\xi(\nabla v)|^4 dx\right]$$
$$\leq 8\left[\frac{8}{\epsilon^4|\xi|^4}\int_D (|\nabla u(x+\epsilon\xi)|^4 + |\nabla u(x)|^4)dx + \frac{8}{\epsilon^4|\xi|^4}\int_D (|\nabla v(x+\epsilon\xi)|^4\right.$$
$$\left. + |\nabla v(x)|^4)dx\right] \leq \frac{128}{\epsilon^4|\xi|^4}(||\nabla u||^4_{L^4(D;\mathbb{R}^{d\times d})} + ||\nabla v||^4_{L^4(D;\mathbb{R}^{d\times d})})$$
$$\leq \frac{128C_e^4}{\epsilon^4|\xi|^4}(||\nabla u||^4_{H^1(D;\mathbb{R}^{d\times d})} + ||\nabla v||^4_{H^1(D;\mathbb{R}^{d\times d})}) \leq \frac{128C_e^4}{\epsilon^4|\xi|^4}(||u||^4_W + ||v||^4_W)$$
$$\leq \frac{128C_e^4}{\epsilon^4|\xi|^4}(||u||_W + ||v||_W)^4. \tag{87}$$

where we used Sobolev embedding property (51) in third last step. Proceeding similarly to get

$$\int_D |S_\xi(\nabla u) - S_\xi(\nabla v)|^4 dx$$
$$\leq \frac{8}{\epsilon^4|\xi|^4}\left[\int_D |\nabla(u-v)(x+\epsilon\xi)|^4 dx + \int_D |\nabla(u-v)(x)|^4 dx\right]$$
$$\leq \frac{16}{\epsilon^4|\xi|^4}||\nabla(u-v)||^4_{L^4(D,\mathbb{R}^{d\times d})}$$
$$\leq \frac{16C_e^4}{\epsilon^4|\xi|^4}||u-v||^4_W. \tag{88}$$

Substituting (87) and (88) into (86) to get

$$\int_D (|S_\xi(\nabla u)| + |S_\xi(\nabla v)|)^2 |S_\xi(\nabla u) - S_\xi(\nabla v)|^2 dx$$
$$\leq \left(\frac{128C_e^4}{\epsilon^4|\xi|^4}(||u||_W + ||v||_W)^4\right)^{1/2}\left(\frac{16C_e^4}{\epsilon^4|\xi|^4}||u-v||^4_W\right)^{1/2}$$
$$= \frac{32\sqrt{2}C_e^4}{\epsilon^4|\xi|^4}(||u||_W + ||v||_W)^2||u-v||^2_W$$
$$\leq \frac{64C_e^4}{\epsilon^4|\xi|^4}(||u||_W + ||v||_W)^2||u-v||^2_W.$$

Substituting above in (85) to get

$$||I_6||^2 \le \left(\frac{2C_3}{\epsilon\omega_d}\right)^2 \omega_d \bar{J}_{3/2} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}} |\xi|^3 \epsilon |\xi| \left[\frac{64C_e^4}{\epsilon^4 |\xi|^4} (||u||_W + ||v||_W)^2 ||u-v||_W^2\right] d\xi.$$

From above we have

$$||I_6|| \le \frac{16C_3C_e^2\bar{J}_{3/2}(||u||_W + ||v||_W)}{\epsilon^{5/2}} ||u-v||_W = \frac{L_4(||u||_W + ||v||_W)}{\epsilon^{5/2}} ||u-v||_W, \tag{89}$$

where we let $L_4 = 16C_3C_e^2\bar{J}_{3/2}$.

From the expression of $h_3(u)(x)$ and $h_5(u)(x)$, we find that it is similar to term $g_1(u)(x)$ from the point of view of L^2 norm. Also, $h_4(u)(x)$ is similar to $P(u)(x)$. We easily have

$$|h_4(u)(x) - h_4(v)(x)| \le \frac{2C_2C_{\omega_2}}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|)|S_\xi(u) - S_\xi(v)| d\xi,$$

where we used the fact that $|\nabla^2\bar{\omega}_\xi(x)| \le C_{\omega_2}$. Above is similar to the bound on $|P(u)(x) - P(v)(x)|$ (see (64)); therefore we have

$$||h_4(u) - h_4(v)|| \le \frac{L_1C_{\omega_2}}{\epsilon^2} ||u-v||_W. \tag{90}$$

Similarly, we have

$$\begin{aligned} |h_3(u)(x) - h_3(v)(x)| &\le \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|)|F_1''(\sqrt{s_\xi} S_\xi(u)) \\ &- F_1''(\sqrt{s_\xi} S_\xi(v))||\nabla\bar{\omega}_\xi(x)||S_\xi(\nabla u)| d\xi + \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|)|F_1''(\sqrt{s_\xi} S_\xi(v))||e_\xi \\ &\otimes \nabla\bar{\omega}_\xi(x) \otimes S_\xi(\nabla u) - e_\xi \otimes \nabla\bar{\omega}_\xi(x) \otimes S_\xi(\nabla v)| d\xi \le \frac{2C_3C_{\omega_1}}{\epsilon\omega_d} \int_{H_1(0)} \\ &J(|\xi|)\sqrt{s_\xi}|S_\xi(u) - S_\xi(v)||S_\xi(\nabla u)| d\xi + \frac{2C_2C_{\omega_1}}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|)|S_\xi(\nabla u) \\ &- S_\xi(\nabla v)| d\xi = C_{\omega_1}(I_1(x) + I_2(x)), \end{aligned} \tag{91}$$

where $I_1(x)$ and $I_2(x)$ are given by (68). From (70) to (71), we have

$$||h_3(u) - h_3(v)|| \le C_{\omega_1}(||I_1|| + ||I_2||) \\ \le \frac{\sqrt{\epsilon} C_{\omega_1} L_1 + C_{\omega_1} L_2(||u||_W + ||v||_W)}{\epsilon^{5/2}} ||u - v||_W. \quad (92)$$

Expression of $h_3(u)$ and $h_5(u)$ is similar and hence we have

$$||h_5(u) - h_5(v)|| \le C_{\omega_1}(||I_1|| + ||I_2||) \\ \le \frac{\sqrt{\epsilon} C_{\omega_1} L_1 + C_{\omega_1} L_2(||u||_W + ||v||_W)}{\epsilon^{5/2}} ||u - v||_W. \quad (93)$$

Collecting our results delivers the bound

$$||\nabla^2 P(u) - \nabla^2 P(v)|| \\ \le \left[\frac{\epsilon L_1 + \sqrt{\epsilon} L_2(||u||_W + ||v||_W) + L_3(||u||_W + ||v||_W)^2 + \sqrt{\epsilon} L_4(||u||_W + ||v||_W)}{\epsilon^3} \right. \\ \left. + \frac{\epsilon C_{\omega_2} L_1 + 2\epsilon C_{\omega_1} L_1 + 2\sqrt{\epsilon} C_{\omega_1} L_2(||u||_W + ||v||_W)}{\epsilon^3} \right] ||u - v||_W \\ \le \left[\frac{\epsilon(1 + 2C_{\omega_1} + C_{\omega_2}) L_1 + \sqrt{\epsilon}(L_2 + 2C_{\omega_1} L_2 + L_4)(||u||_W + ||v||_W)}{\epsilon^3} \right. \\ \left. + \frac{L_3(||u||_W + ||v||_W)^2}{\epsilon^3} \right] ||u - v||_W. \quad (94)$$

We now combine (65), (67), (75), and (94) to get

$$||P(u) - P(v)||_W \\ \le \left[\frac{2\epsilon L_1 + \epsilon(1 + C_{\omega_1}) L_1 + \sqrt{\epsilon}(||u||_W + ||v||_W)}{\epsilon^3} \right. \\ + \frac{\epsilon(1 + 2C_{\omega_1} + C_{\omega_2}) L_1 + \sqrt{\epsilon}(L_2 + 2C_{\omega_1} L_2 + L_4)(||u||_W + ||v||_W)}{\epsilon^3} \\ \left. + \frac{L_3(||u||_W + ||v||_W)^2}{\epsilon^3} \right] ||u - v||_W. \quad (95)$$

Finally we let

$$\bar{L}_1 := (4 + 3C_{\omega_1} + C_{\omega_2}) L_1, \ \bar{L}_2 := (1 + 2C_{\omega_1}) L_2 + L_4, \ \bar{L}_3 := L_3 \quad (96)$$

and write

$$
\begin{aligned}
&||P(u) - P(v)||_W \\
&\leq \frac{\bar{L}_1 + \bar{L}_2(||u||_W + ||v||_W) + \bar{L}_3(||u||_W + ||v||_W)^2}{\epsilon^3}||u - v||_W. \qquad (97)
\end{aligned}
$$

This completes the proof of (55).

We now obtain an upper bound on the peridynamic force. Note that $F_1'(0) = 0$ and $S_\xi(v) = 0$ if $v = 0$. Substituting $v = 0$ in (65) and (67) to get

$$
||P(u)|| + ||P(u)||_\infty \leq \frac{2L_1}{\epsilon^2}||u||_W. \qquad (98)
$$

For $||g_1(u)||$ and $||g_2(u)||$ we proceed differently. For $||g_2(u)||$, we substitute $v = 0$ in (74) to get

$$
||g_2(u)|| \leq \frac{C_{\omega_1} L_1}{\epsilon^2}||u||_W. \qquad (99)
$$

To estimate $||g_1(u)||$, we first estimate

$$
\begin{aligned}
|g_1(u)(x)| &\leq \frac{2C_2}{\epsilon\omega_d}\int_{H_1(0)} J(|\xi|)|\nabla S_\xi(u)|d\xi \\
&\leq \frac{2C_2}{\epsilon^2\omega_d}\int_{H_1(0)} \frac{J(|\xi|)}{|\xi|}(|\nabla u(x+\epsilon\xi)| + |\nabla u(x)|)d\xi, \qquad (100)
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
||g_1(u)||^2 &\leq \left(\frac{2C_2}{\epsilon^2\omega_d}\right)^2 \omega_d \bar{J}_1 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|}\left[\int_D (|\nabla u(x+\epsilon\xi)| + |\nabla u(x)|)^2 dx\right] d\xi \\
&\leq \left(\frac{4C_2\bar{J}_1}{\epsilon^2}\right)^2 ||\nabla u||^2 \qquad (101)
\end{aligned}
$$

i.e.

$$
||g_1(u)|| \leq \frac{L_1}{\epsilon^2}||u||_W. \qquad (102)
$$

Combining (99) and (102) gives

$$
||\nabla P(u)|| \leq \frac{(1 + C_{\omega_1})L_1}{\epsilon^2}||u||_W. \qquad (103)
$$

We need to estimate $||\nabla^2 P(u)||$. We have from (63)

$$||\nabla^2 P(u)|| \leq ||h_1(u)|| + ||h_2(u)|| + ||h_3(u)|| + ||h_4(u)|| + ||h_5(u)||.$$

From the expression of $h_1(u)$ and $h_2(u)$, we find that

$$||h_1(u)|| \leq \frac{4C_2\bar{J}_1}{\epsilon^2}||u||_W = \frac{L_1}{\epsilon^2}||u||_W \text{ and } ||h_2(u)|| \leq \frac{8C_3C_e^2\bar{J}_{3/2}}{\epsilon^{5/2}}||u||_W^2 \leq \frac{L_4}{\epsilon^{5/2}}||u||_W^2,$$

where $L_4 = 16C_3C_e^2\bar{J}_{3/2}$. Case of $||h_3(u)||$ and $||h_5(u)||$ is similar to $||g_1(u)||$, and case of $||h_4(u)||$ is similar to $||P(u)||$.

$$||h_4(u)|| \leq \frac{C_{\omega_2}L_1}{\epsilon^2}||u||_W$$

and

$$||h_3(u)|| \leq \frac{C_{\omega_1}L_1}{\epsilon^2}||u||_W \quad \text{and} \quad ||h_5(u)|| \leq \frac{C_{\omega_1}L_1}{\epsilon^2}||u||_W.$$

We combine the inequalities listed above to get

$$||\nabla^2 P(u)|| \leq \frac{\sqrt{\epsilon}(1 + C_{\omega_2} + 2C_{\omega_1})L_1 + L_4||u||_W}{\epsilon^{5/2}}||u||_W. \tag{104}$$

Finally, after combining (98), (103), and (104), we get

$$||P(u)||_W \leq \frac{\sqrt{\epsilon}(4 + 3C_{\omega_1} + C_{\omega_2})L_1 + L_4||u||_W}{\epsilon^{5/2}}||u||_W.$$

We let

$$\bar{L}_4 := \bar{L}_1 \qquad \text{and} \qquad \bar{L}_5 := L_4 \tag{105}$$

to write

$$||P(u)||_W \leq \frac{\bar{L}_4||u||_W + \bar{L}_5||u||_W^2}{\epsilon^{5/2}}. \tag{106}$$

This completes the proof of (56) and this completes the proof of Theorem 3.

Local and Global Existence of Solution in $H^2 \cap L^\infty$ Space

We now prove Theorem 4. We first prove local existence for a finite time interval. We find that we can choose this time interval independent of the initial data. We repeat the local existence theorem to uniquely continue the local solution over any finite time interval. The existence and uniqueness of local solutions is stated in the following theorem.

Theorem 5 (Local existence and uniqueness). *Given $X = W \times W$, $b(t) \in W$, and initial data $x_0 = (u_0, v_0) \in X$. We suppose that $b(t)$ is continuous in time over some time interval $I_0 = (-T, T)$ and satisfies $\sup_{t \in I_0} ||b(t)||_W < \infty$. Then, there exists a time interval $I' = (-T', T') \subset I_0$ and unique solution $y = (y^1, y^2)$ such that $y \in C^1(I'; X)$ and*

$$y(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau)\, d\tau, \text{ for } t \in I' \tag{107}$$

or equivalently

$$y'(t) = F^\epsilon(y(t), t), \text{with } y(0) = x_0, \text{ for } t \in I'$$

where $y(t)$ and $y'(t)$ are Lipschitz continuous in time for $t \in I' \subset I_0$.

Proof. To prove Theorem 5, we proceed as follows. Write $y(t) = (y^1(t), y^2(t))$ with $||y||_X = ||y^1(t)||_W + ||y^2(t)||_W$. Let us consider $R > ||x_0||_X$ and define the ball $B(0, R) = \{y \in X : ||y||_X < R\}$. Let $r < \min\{1, R - ||x_0||_X\}$. We clearly have $r^2 < (R - ||x_0||_X)^2$ as well as $r^2 < r < R - ||x_0||_X$. Consider the ball $B(x_0, r^2)$ defined by

$$B(x_0, r^2) = \{y \in X : ||y - x_0||_X < r^2\}. \tag{108}$$

Then we have $B(x_0, r^2) \subset B(x_0, r) \subset B(0, R)$ (see Fig. 7).

To recover the existence and uniqueness, we introduce the transformation

$$S_{x_0}(y)(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau)\, d\tau.$$

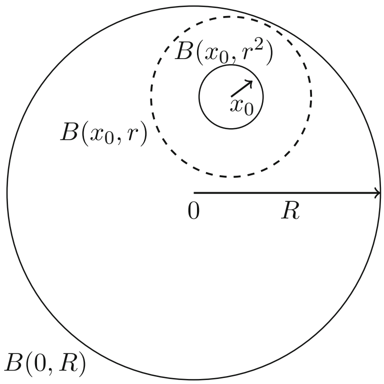

Fig. 7 Geometry

Introduce $0 < T' < T$ and the associated set $Y(T')$ of functions in W taking values in $B(x_0, r^2)$ for $I' = (-T', T') \subset I_0 = (-T, T)$. The goal is to find appropriate interval $I' = (-T', T')$ for which S_{x_0} maps into the corresponding set $Y(T')$. Writing out the transformation with $y(t) \in Y(T')$ gives

$$S^1_{x_0}(y)(t) = x^1_0 + \int_0^t y^2(\tau)\, d\tau \tag{109}$$

$$S^2_{x_0}(y)(t) = x^2_0 + \int_0^t (-\nabla PD^\epsilon(y^1(\tau)) + b(\tau))\, d\tau. \tag{110}$$

We have from (109)

$$||S^1_{x_0}(y)(t) - x^1_0||_W \leq \sup_{t \in (-T', T')} ||y^2(t)||_W T'. \tag{111}$$

Using bound on $-\nabla PD^\epsilon$ in Theorem 3, we have from (110)

$$||S^2_{x_0}(y)(t) - x^2_0||_W \leq \int_0^t \left[\frac{\bar{L}_4}{\epsilon^{5/2}} ||y^1(\tau)||_W + \frac{\bar{L}_5}{\epsilon^{5/2}} ||y^1(\tau)||^2_W + ||b(\tau)||_W \right] d\tau. \tag{112}$$

Let $\bar{b} = \sup_{t \in I_0} ||b(t)||_W$. Noting that transformation S_{x_0} is defined for $t \in I' = (-T', T')$ and $y(\tau) = (y^1(\tau), y^2(\tau)) \in B(x_0, r^2) \subset B(0, R)$ as $y \in Y(T')$, we have from (112) and (111)

$$||S^1_{x_0}(y)(t) - x^1_0||_W \leq RT',$$

$$||S^2_{x_0}(y)(t) - x^2_0||_W \leq \left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + \bar{b} \right] T'.$$

Adding gives

$$||S_{x_0}(y)(t) - x_0||_X \leq \left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b} \right] T'. \tag{113}$$

Choosing T' as below

$$T' < \frac{r^2}{\left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b} \right]} \tag{114}$$

will result in $S_{x_0}(y) \in Y(T')$ for all $y \in Y(T')$ as

$$||S_{x_0}(y)(t) - x_0||_X < r^2. \tag{115}$$

Since $r^2 < (R - ||x_0||_X)^2$, we have

$$T' < \frac{r^2}{\left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b}\right]} < \frac{(R - ||x_0||_X)^2}{\left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b}\right]}.$$

Let $\theta(R)$ be given by

$$\theta(R) := \frac{(R - ||x_0||_X)^2}{\left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b}\right]}. \tag{116}$$

$\theta(R)$ is increasing with $R > 0$ and satisfies

$$\theta_\infty := \lim_{R \to \infty} \theta(R) = \frac{\epsilon^{5/2}}{\bar{L}_5}. \tag{117}$$

So given R and $||x_0||_X$, we choose T' according to

$$\frac{\theta(R)}{2} < T' < \theta(R), \tag{118}$$

and set $I' = (-T', T')$. This way we have shown that for time domain I' the transformation $S_{x_0}(y)(t)$ as defined in Eqs. 109 and 110 maps $Y(T')$ into itself. Existence and uniqueness of solution can be established using (Theorem 6.10, Driver 2003). □

We now prove Theorem 1. From the proof of Theorem 2 above, we have a unique local solution over a time domain $(-T', T')$ with $\frac{\theta(R)}{2} < T'$. Since $\theta(R) \nearrow \epsilon^{5/2}/\bar{L}_5$ as $R \nearrow \infty$, we can fix a tolerance $\eta > 0$ so that $[(\epsilon^{5/2}/2\bar{L}_5) - \eta] > 0$. Then for any initial condition in W and $b = \sup_{t \in [-T,T)} ||b(t)||_W$, we can choose R sufficiently large so that $||x_0||_X < R$ and $0 < (\epsilon^{5/2}/2\bar{L}_5) - \eta < T'$. Since choice of T' is independent of initial condition and R, we can always find local solutions for time intervals $(-T', T')$ for T' larger than $[(\epsilon^{5/2}/2\bar{L}_5) - \eta] > 0$. Therefore we apply the local existence and uniqueness result to uniquely continue local solutions up to an arbitrary time interval $(-T, T)$.

Conclusions: Convergence of Regular Solutions in the Limit of Vanishing Horizon

In this final section, we examine the behavior of bounded Hölder continuous solutions as the peridynamic horizon tends to zero. We find that the solutions converge to a limiting sharp fracture evolution with bounded Griffiths fracture energy and satisfy the linear elastic wave equation away from the fracture set. We look at a subset of Hölder solutions that are differentiable in the spatial variables to show that sharp fracture evolutions can be approached by spatially smooth evolutions in the limit of vanishing nonlocality. As ϵ approaches zero, derivatives can become large but must localize to surfaces across which the limiting evolution jumps. These conclusions are reported in Jha and Lipton (2017b). The same behavior can be recovered for bounded H^2 solutions in the limit of vanishing horizon. These results support the numerical simulation for more regular nonlocal evolutions that approximate sharp fracture in the limit of vanishing nonlocality. In the next chapter we provide a priori estimates of the finite difference and finite element approximations to fracture evolution for nonlocal models with horizon $\epsilon > 0$.

To fix ideas consider a sequence of peridynamic horizons $\epsilon_k = 1/k, k = 1, \dots$ and the associated Hölder continuous solutions $\boldsymbol{u}^{\epsilon_k}(t, \boldsymbol{x})$ of the peridynamic initial value problem Eqs. 1, 2, and 3. We assume that the initial conditions $\boldsymbol{u}_0^{\epsilon_k}, \boldsymbol{v}_0^{\epsilon_k}$ have uniformly bounded peridynamic energy and mean square initial velocity given by

$$\sup_{\epsilon_k} PD^{\epsilon_k}(\boldsymbol{u}_0^{\epsilon_k}) < \infty \text{ and } \sup_{\epsilon_k} ||\boldsymbol{v}_0^{\epsilon_k}||_{L^2(D;\mathbb{R}^d)} < \infty.$$

Moreover we suppose that $\boldsymbol{u}_0^{\epsilon_k}, \boldsymbol{v}_0^{\epsilon_k}$ are differentiable on D and that they converge in $L^2(D;\mathbb{R})$ to $\boldsymbol{u}_0^0, \boldsymbol{v}_0^0$ with bounded Griffith free energy given by

$$\int_D 2\mu|\mathcal{E}\boldsymbol{u}_0^0|^2 + \lambda|\text{div}\,\boldsymbol{u}_0^0|^2\,dx + \mathcal{G}_c\mathcal{H}^{d-1}(J_{\boldsymbol{u}_0^0}) \le C < \infty,$$

where $J_{\boldsymbol{u}_0^0}$ denotes an initial fracture surface given by the jumps in the initial deformation $\boldsymbol{u}_0^0$ and $\mathcal{H}^2(J_{\boldsymbol{u}^0(t)})$ is its *two*-dimensional Hausdorff measure of the jump set. Here $\mathcal{E}\boldsymbol{u}_0^0$ is the elastic strain and $\text{div}\,\boldsymbol{u}_0^0 = Tr(\mathcal{E}\boldsymbol{u}_0^0)$. The constants μ, λ are given by the explicit formulas

$$\text{and } \mu = \lambda = \tfrac{1}{5}f'(0)\int_0^1 r^d J(r)dr,\ d = 2, 3$$

and

$$\mathcal{G}_c = \frac{3}{2}f_\infty\int_0^1 r^d J(r)dr,\ d = 2, 3,$$

where $f'(0)$ and f_∞ are defined by Eq. 6. Here $\mu = \lambda$ and is a consequence of the central force model used in cohesive dynamics. Last we suppose as in Lipton (2016) that the solutions are uniformly bounded, i.e.,

$$\sup_{\epsilon_k}\sup_{[0,T]} ||\boldsymbol{u}^{\epsilon_k}(t)||_{L^\infty(D;\mathbb{R}^d)} < \infty,$$

The Hölder solutions $\boldsymbol{u}^{\epsilon_k}(t, \boldsymbol{x})$ naturally belong to $L^2(D;\mathbb{R}^d)$ for all $t \in [0, T]$, and we can directly apply the Gronwall inequality (Equation (6.9) of Lipton

2016) together with Theorems 6.2 and 6.4 of Lipton (2016) to conclude similar to Theorems 5.1 and 5.2 of Lipton (2016) that there is at least one "cluster point" $\boldsymbol{u}^0(t, \boldsymbol{x})$ belonging to $C([0, T]; L^2(D; \mathbb{R}^d))$ and subsequence, also denoted by $\boldsymbol{u}^{\epsilon_k}(t, \boldsymbol{x})$ for which

$$\lim_{\epsilon_k \to 0} \max_{0 \leq t \leq T} \left\{ \|\boldsymbol{u}^{\epsilon_k}(t) - \boldsymbol{u}^0(t)\|_{L^2(D;\mathbb{R}^d)} \right\} = 0.$$

Moreover it follows from Lipton (2016) that the limit evolution $\boldsymbol{u}^0(t, \boldsymbol{x})$ has a weak derivative $\boldsymbol{u}_t^0(t, \boldsymbol{x})$ belonging to $L^2([0, T] \times D; \mathbb{R}^d)$. For each time $t \in [0, T]$, we can apply methods outlined in Lipton (2016) to find that the cluster point $\boldsymbol{u}^0(t, \boldsymbol{x})$ is a special function of bounded deformation (see Ambrosio et al. 1997; Bellettini et al. 1998) and has bounded linear elastic fracture energy given by

$$\int_D 2\mu |\mathcal{E}\boldsymbol{u}^0(t)|^2 + \lambda |\text{div}\, \boldsymbol{u}^0(t)|^2 \, dx + \mathcal{G}_c \mathcal{H}^2(J_{\boldsymbol{u}^0(t)}) \leq C,$$

for $0 \leq t \leq T$ where $J_{\boldsymbol{u}^0(t)}$ denotes the evolving fracture surface The deformation-crack set pair $(\boldsymbol{u}^0(t), J_{\boldsymbol{u}^0(t)})$ records the brittle fracture evolution of the limit dynamics.

Arguments identical to Lipton (2016) show that away from sets where $|S(\boldsymbol{y}, \boldsymbol{x}; \boldsymbol{u}^{\epsilon_k})| > S_c$, the limit $\boldsymbol{u}^0$ satisfies the linear elastic wave equation. This is stated as follows: Fix $\delta > 0$ and for $\epsilon_k < \delta$ and $0 \leq t \leq T$ consider the open set $D' \subset D$ for which points $\boldsymbol{x}$ in D' and $\boldsymbol{y}$ for which $|\boldsymbol{y} - \boldsymbol{x}| < \epsilon_k$ satisfy,

$$|S(\boldsymbol{y}, \boldsymbol{x}; \boldsymbol{u}^{\epsilon_k}(t))| < S_c(\boldsymbol{y}, \boldsymbol{x}).$$

Then the limit evolution $\boldsymbol{u}^0(t, \boldsymbol{x})$ evolves elastodynamically on D' and is governed by the balance of linear momentum expressed by the Navier-Lamé equations on the domain $[0, T] \times D'$ given by

$$\boldsymbol{u}_{tt}^0(t) = \text{div}\boldsymbol{\sigma}(t) + \boldsymbol{b}(t), \text{on } [0, T] \times D',$$

where the stress tensor $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = \lambda I_d Tr(\mathcal{E}\, \boldsymbol{u}^0) + 2\mu \mathcal{E} \boldsymbol{u}^0,$$

where I_d is the identity on $\mathbb{R}^d$ and $Tr(\mathcal{E}\, \boldsymbol{u}^0)$ is the trace of the strain. Here the second derivative $\boldsymbol{u}_{tt}^0$ is the time derivative in the sense of distributions of $\boldsymbol{u}_t^0$, and $\text{div}\boldsymbol{\sigma}$ is the divergence of the stress tensor $\boldsymbol{\sigma}$ in the distributional sense. This shows that sharp fracture evolutions can be approached by spatially smooth evolutions in the limit of vanishing nonlocality.

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