

## NUMERICAL ANALYSIS OF NONLOCAL FRACTURE MODELS IN HÖLDER SPACE\*

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**Abstract.** In this work, we calculate the convergence rate of the finite difference approximation for a class of nonlocal fracture models. We consider two point force interactions characterized by a double well potential. We show the existence of a evolving displacement field in Hölder space with Hölder exponent  $\gamma \in (0, 1]$ . The rate of convergence of the finite difference approximation depends on the factor  $C_s h^\gamma / \epsilon^2$  where  $\epsilon$  gives the length scale of nonlocal interaction,  $h$  is the discretization length, and  $C_s$  is the maximum of Hölder norm of the solution and its second derivatives during the evolution. It is shown that the rate of convergence holds for both the forward Euler scheme as well as general single step implicit schemes. A stability result is established for the semidiscrete approximation. The Hölder continuous evolutions are seen to converge to a brittle fracture evolution in the limit of vanishing nonlocality.

**Key words.** nonlocal fracture models, peridynamics, cohesive dynamics, numerical analysis, finite difference approximation

**AMS subject classifications.** 34A34, 34B10, 74H55, 74S20

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**1. Introduction.** Nonlocal formulations have been proposed to describe the evolution of deformations which exhibit loss of differentiability and continuity; see [28] and [31]. These models are commonly referred to as peridynamic models. The main idea is to define the strain in terms of displacement differences and allow nonlocal interactions between material points. This generalization of strain allows for the participation of a larger class of deformations in the dynamics. Numerical simulations based on peridynamic modeling exhibit formation and evolution of sharp interfaces associated with phase transformation and fracture [9], [32], [27], [15], [1], [12], [24], [6], [19], [30], [34], [16]. A recent summary of the state of the art can be found in [14].

In this work, we provide a numerical analysis for the class of nonlocal models introduced in [22] and [23]. These models are defined by a double well two point potential. Here one potential well is centered at zero and associated with elastic response while the other well is at infinity and associated with surface energy. The rationale for studying these models is that they are shown to be well posed over the class of square integrable nonsmooth displacements and, in the limit of vanishing nonlocality, their dynamics recover features associated with sharp fracture propagation; see [22] and [23]. The numerical simulation of prototypical fracture problems using this model is carried out in [24]. In order to develop an  $L^2$  approximation theory, we show the nonlocal evolution is well posed over a more regular space of functions. To include displacement fields which have no well-defined derivatives, we consider displacement fields in the Hölder space  $C^{0,\gamma}$  with Hölder exponent  $\gamma$  taking any value in  $(0, 1]$ . We show that a unique evolution exists in  $C^{0,\gamma}$  for  $C^{0,\gamma}$  initial data and body force. The semidiscrete approximation to the Hölder continuous evolution is considered and it is

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shown that at any time its energy is bounded by the initial energy and the work done by the body force. We develop an approximation theory for the forward Euler scheme and show that these ideas can be easily extended to the backward Euler scheme as well as other implicit one step time discretization schemes. It is found that the discrete approximation converges to the exact solution in the  $L^2$  norm uniformly over finite time intervals with the rate of convergence proportional to  $(C_t \Delta t + C_s h^\gamma / \epsilon^2)$ , where  $\Delta t$  is the size of time step,  $h$  is the size of spatial mesh discretization, and  $\epsilon$  is the length scale of nonlocal interaction relative to the size of the domain. The constant  $C_t$  depends on the  $L^2$  norm of the time derivatives of the solution and  $C_s$  depends on the Hölder norm of the solution and the Lipschitz constant of peridynamic force. We point out that the constants appearing in the convergence estimates with respect to  $h$  can be dependent on the horizon and be large when  $\epsilon$  is small. This is discussed in section 3 and an example is provided in section 6. These results show that while errors can grow with each time step, they can be controlled over finite times  $t$  by suitable spatial temporal mesh refinement. We then apply the methods developed in [22] and [23] to show that in the limit  $\epsilon \rightarrow 0$ , the Hölder continuous evolutions converge to a limiting sharp fracture evolution with bounded Griffith's fracture energy. Here the limit evolution is differentiable off the crack set and satisfies the linear elastic wave equation.

In the language of nonlocal operators, the integral kernel associated with the nonlocal model studied here is Lipschitz continuous guaranteeing global stability of the finite difference approximation. This is in contrast to PDE based evolutions where stability can be conditional. In addition we examine local stability. Unfortunately the problem is nonlinear so we don't establish Courant–Friedrichs–Lewy (CFL) conditions but instead identify a mode of dynamic instability that can arise during the evolution. This type of instability is due to a radial perturbation of the solution and causes error to grow with each time step for the Euler scheme. For implicit schemes this perturbation can become unstable in parts of the computational domain where there is material softening; see subsection 3.4. Of course stability conditions like the CFL conditions for linear nonlocal equations are of importance for guidance in implementations. In the case of  $d = 1$ , a CFL type condition is obtained for the finite difference and finite element approximation of the linear peridynamic equation; see [17]. Recent work develops a new simple CFL condition for one dimensional linearized peridynamics in the absence of body forces [20]. Related analysis for the linear peridynamic equation in one dimension is taken up in [35] and [7]. The recent and related work [13] and [18] addresses numerical approximation for problems of nonlocal diffusion.

There is now a large body of contemporary work addressing the numerical approximation of singular kernels with application to nonlocal diffusion, advection, and mechanics. Numerical formulations and convergence theory for nonlocal  $p$ -Laplacian formulations are developed in [10] and [26]. Numerical analysis of nonlocal steady state diffusion is presented in [33], [25], and [8]. The use of fractional Sobolev spaces for nonlocal problems is investigated and developed in [13]. Quadrature approximations and stability conditions for linear peridynamics are analyzed in [35] and [29]. The interplay between nonlocal interaction length and grid refinement for linear peridynamic models is presented in [7]. Analysis of adaptive refinement and domain decomposition for linearized peridynamics are provided in [3], [21], and [2]. This list is by no means complete and the literature on numerical methods and analysis continues to grow.

The paper is organized as follows. In section 2, we describe the nonlocal model.

In subsection 2.2, we state theorems which show Lipschitz continuity of the nonlocal force (Theorem 1) and the existence and uniqueness of an evolution over any finite time interval (Theorem 2). In section 3, we compute the convergence rate of the forward Euler scheme as well as implicit one step methods. We identify stability of the semidiscrete approximation with respect to the energy in subsection 3.3. In subsection 3.4, we identify local instabilities in the fully discrete evolution caused by suitable radial perturbations of the solution. In section 4, we give the proof of Theorems 1, 6, and 2. The convergence of Hölder continuous evolutions to sharp fracture evolutions as  $\epsilon \rightarrow 0$  is shown in section 5. In section 6, we present an example showing the effect of the constants  $C_t$  and  $C_s$  on the convergence rate and summarize our results.

**2. Double well potential and existence of a solution.** In this section, we present the nonlinear nonlocal model. Let  $D \subset \mathbb{R}^d$ ,  $d = 2, 3$ , be the material domain with characteristic length-scale of unity. Let  $\epsilon \in (0, 1]$  be the size of horizon across which nonlocal interaction between points takes place. The material point  $\mathbf{x} \in D$  interacts nonlocally with all material points inside a horizon of length  $\epsilon$ . Let  $H_\epsilon(\mathbf{x})$  be the ball of radius  $\epsilon$  centered at  $\mathbf{x}$  containing all points  $\mathbf{y}$  that interact with  $\mathbf{x}$ . After deformation the material point  $\mathbf{x}$  assumes position  $\mathbf{z} = \mathbf{x} + \mathbf{u}(\mathbf{x})$ . In this treatment we assume infinitesimal displacements and the strain is written in terms of the displacement  $\mathbf{u}$  as

$$S = S(\mathbf{y}, \mathbf{x}; \mathbf{u}) := \frac{\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|} \cdot \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|}.$$

Let  $W^\epsilon(S, \mathbf{y} - \mathbf{x})$  be the nonlocal potential density per unit length between material point  $\mathbf{y}$  and  $\mathbf{x}$ . The energy density at  $\mathbf{x}$  is given by

$$\mathbf{W}^\epsilon(S, \mathbf{x}) = \frac{1}{\epsilon^d \omega_d} \int_{H_\epsilon(\mathbf{x})} |\mathbf{y} - \mathbf{x}| W^\epsilon(S, \mathbf{y} - \mathbf{x}) d\mathbf{y},$$

where  $\omega_d$  is the volume of a unit ball in  $d$ -dimension and  $\epsilon^d \omega_d$  is the volume of the ball of radius  $\epsilon$ . The potential energy is written as

$$PD^\epsilon(\mathbf{u}) = \int_D \mathbf{W}^\epsilon(S(\mathbf{u}), \mathbf{x}) d\mathbf{x},$$

and the displacement field satisfies the following equation of motion:

$$(1) \quad \rho \partial_{tt}^2 \mathbf{u}(t, \mathbf{x}) = -\nabla PD^\epsilon(\mathbf{u}) + \mathbf{b}(t, \mathbf{x})$$

for all  $\mathbf{x} \in D$ . Here we have

$$-\nabla PD^\epsilon(\mathbf{u})(\mathbf{x}) = \frac{2}{\epsilon^d \omega_d} \int_{H_\epsilon(\mathbf{x})} \partial_S W^\epsilon(S, \mathbf{y} - \mathbf{x}) \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y},$$

where  $\mathbf{b}(t, \mathbf{x})$  is the body force,  $\rho$  is the density, and  $\partial_S W^\epsilon$  is the derivative of potential with respect to the strain.

We prescribe the zero Dirichlet condition on the boundary of  $D$ ,

$$(2) \quad \mathbf{u}(\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \partial D,$$

where we have denoted the boundary by  $\partial D$ . We extend the zero boundary condition outside  $D$  to  $\mathbb{R}^3$ .

The peridynamic equation, boundary conditions, and initial conditions

$$(3) \quad \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \partial_t \mathbf{u}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x})$$

determine the peridynamic evolution  $\mathbf{u}(t, \mathbf{x})$ .

*Peridynamics energy.* The total energy  $\mathcal{E}^\epsilon(\mathbf{u})(t)$  is given by the sum of kinetic and potential energy given by

$$(4) \quad \mathcal{E}^\epsilon(\mathbf{u})(t) = \frac{1}{2} \|\dot{\mathbf{u}}(t)\|_{L^2(D; \mathbb{R}^d)}^2 + PD^\epsilon(\mathbf{u}(t)),$$

where potential energy  $PD^\epsilon$  is given by

$$PD^\epsilon(\mathbf{u}) = \int_D \left[ \frac{1}{\epsilon^d \omega_d} \int_{H_\epsilon(\mathbf{x})} W^\epsilon(S(\mathbf{u}), \mathbf{y} - \mathbf{x}) d\mathbf{y} \right] d\mathbf{x}.$$

Differentiation of (4) gives the identity

$$(5) \quad \frac{d}{dt} \mathcal{E}^\epsilon(\mathbf{u})(t) = (\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) - (-\nabla PD^\epsilon(\mathbf{u}(t)), \dot{\mathbf{u}}(t)),$$

where  $(\cdot, \cdot)$  is the inner product on  $L^2(\mathbb{R}^d, D)$  and  $\|\cdot\|_{L^2(\mathbb{R}^d, D)}$  is the associated norm.

**2.1. Nonlocal potential.** We consider the nonlocal two point interaction potential density  $W^\epsilon$  of the form

$$(6) \quad W^\epsilon(S, \mathbf{y} - \mathbf{x}) = \omega(\mathbf{x})\omega(\mathbf{y}) \frac{J^\epsilon(|\mathbf{y} - \mathbf{x}|)}{\epsilon |\mathbf{y} - \mathbf{x}|} f(|\mathbf{y} - \mathbf{x}| S^2),$$

where  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  is assumed to be positive, smooth, and concave with the following properties:

$$(7) \quad \lim_{r \rightarrow 0^+} \frac{f(r)}{r} = f'(0), \quad \lim_{r \rightarrow \infty} f(r) = f_\infty < \infty.$$

The potential  $W^\epsilon(S, \mathbf{y} - \mathbf{x})$  is of double well type and convex near the origin where it has one well and is concave and bounded at infinity where it has the second well.  $J^\epsilon(|\mathbf{y} - \mathbf{x}|)$  models the influence of separation between points  $\mathbf{y}$  and  $\mathbf{x}$ . We define  $J^\epsilon$  by rescaling  $J(|\boldsymbol{\xi}|)$ , i.e.,  $J^\epsilon(|\boldsymbol{\xi}|) = J(|\boldsymbol{\xi}|/\epsilon)$ . Here  $J$  is zero outside the ball  $H_1(\mathbf{0})$  and satisfies  $0 \leq J(|\boldsymbol{\xi}|) \leq M$  for all  $\boldsymbol{\xi} \in H_1(\mathbf{0})$ . The domain function  $\omega$  enforces boundary conditions on  $\partial_S W^\epsilon$  at the boundary of the body  $D$ . Here the boundary is denoted by  $\partial D$  and  $\omega$  is a nonnegative differentiable function  $0 \leq \omega \leq 1$ . On the boundary  $\omega = 0$  and  $\omega = 1$  for points  $\mathbf{x}$  inside  $D$  with distance greater than  $\epsilon$  away from the boundary. We continue  $\omega$  by zero for all points outside  $D$ .

The potential described in (6) gives the convex-concave dependence of  $W(S, \mathbf{y} - \mathbf{x})$  on the strain  $S$  for fixed  $\mathbf{y} - \mathbf{x}$ ; see Figure 1. Initially the force is elastic for small strains and then softens as the strain becomes larger; see Figure 2. The critical strain where the force between  $\mathbf{x}$  and  $\mathbf{y}$  begins to soften is given by  $S_c(\mathbf{y}, \mathbf{x}) := \bar{r}/\sqrt{|\mathbf{y} - \mathbf{x}|}$  and the force decreases monotonically for

$$|S(\mathbf{y}, \mathbf{x}; \mathbf{u})| > S_c.$$

Here  $\bar{r}$  is the inflection point of  $r \rightarrow f(r^2)$  and is the root of the following equation:

$$f'(r^2) + 2r^2 f''(r^2) = 0.$$

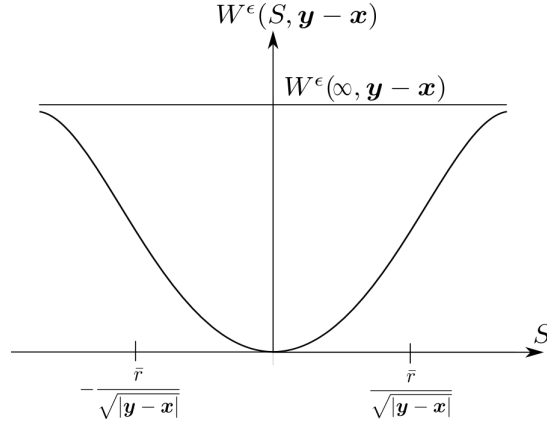


FIG. 1. Two point potential  $W^\epsilon(S, \mathbf{y} - \mathbf{x})$  as a function of strain  $S$  for fixed  $\mathbf{y} - \mathbf{x}$ .

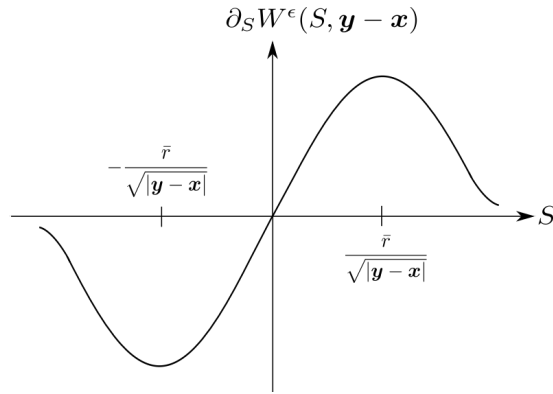


FIG. 2. Nonlocal force  $\partial_S W^\epsilon(S, \mathbf{y} - \mathbf{x})$  as a function of strain  $S$  for fixed  $\mathbf{y} - \mathbf{x}$ . Second derivative of  $W^\epsilon(S, \mathbf{y} - \mathbf{x})$  is zero at  $\pm\bar{r}/\sqrt{|\mathbf{y} - \mathbf{x}|}$ .

**2.2. Existence of solution.** Let  $C^{0,\gamma}(D; \mathbb{R}^d)$  be the Hölder space with exponent  $\gamma \in (0, 1]$ . The closure of continuous functions with compact support on  $D$  in the supremum norm is denoted by  $C_0(D)$ . We identify functions in  $C_0(D)$  with their unique continuous extensions to  $\bar{D}$ . It is easily seen that functions belonging to this space take the value zero on the boundary of  $D$ ; see, e.g., [11]. We introduce  $C_0^{0,\gamma}(D) = C^{0,\gamma}(D) \cap C_0(D)$ . In this paper we extend all functions in  $C_0^{0,\gamma}(D)$  by zero outside  $D$ . The norm of  $\mathbf{u} \in C_0^{0,\gamma}(D; \mathbb{R}^d)$  is taken to be

$$\|\mathbf{u}\|_{C^{0,\gamma}(D; \mathbb{R}^d)} := \sup_{\mathbf{x} \in D} |\mathbf{u}(\mathbf{x})| + [\mathbf{u}]_{C^{0,\gamma}(D; \mathbb{R}^d)},$$

where  $[\mathbf{u}]_{C^{0,\gamma}(D; \mathbb{R}^d)}$  is the Hölder seminorm and given by

$$[\mathbf{u}]_{C^{0,\gamma}(D; \mathbb{R}^d)} := \sup_{\substack{\mathbf{x} \neq \mathbf{y}, \\ \mathbf{x}, \mathbf{y} \in D}} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\gamma},$$

and  $C_0^{0,\gamma}(D; \mathbb{R}^d)$  is a Banach space with this norm. Here we make the hypothesis that the domain function  $\omega$  belongs to  $C_0^{0,\gamma}(D; \mathbb{R}^d)$ .

We write the evolution equation (1) as an equivalent first order system with  $y_1(t) = \mathbf{u}(t)$  and  $y_2(t) = \mathbf{v}(t)$  with  $\mathbf{v}(t) = \partial_t \mathbf{u}(t)$ . Let  $y = (y_1, y_2)^T$  where  $y_1, y_2 \in C_0^{0,\gamma}(D; \mathbb{R}^d)$ , and let  $F^\epsilon(y, t) = (F_1^\epsilon(y, t), F_2^\epsilon(y, t))^T$  such that

$$(8) \quad F_1^\epsilon(y, t) := y_2,$$

$$(9) \quad F_2^\epsilon(y, t) := -\nabla PD^\epsilon(y_1) + \mathbf{b}(t).$$

The initial boundary value associated with the evolution equation (1) is equivalent to the initial boundary value problem for the first order system given by

$$(10) \quad \frac{d}{dt}y = F^\epsilon(y, t),$$

with initial condition given by  $y(0) = (\mathbf{u}_0, \mathbf{v}_0)^T \in C_0^{0,\gamma}(D; \mathbb{R}^d) \times C_0^{0,\gamma}(D; \mathbb{R}^d)$ .

The function  $F^\epsilon(y, t)$  satisfies the Lipschitz continuity given by the following theorem.

**THEOREM 1** (Lipschitz continuity and bound). *Let  $X = C_0^{0,\gamma}(D; \mathbb{R}^d) \times C_0^{0,\gamma}(D; \mathbb{R}^d)$ . The function  $F^\epsilon(y, t) = (F_1^\epsilon, F_2^\epsilon)^T$ , as defined in (8) and (9), is Lipschitz continuous in any bounded subset of  $X$ . We have, for any  $y, z \in X$  and  $t > 0$ ,*

$$(11) \quad \begin{aligned} & \|F^\epsilon(y, t) - F^\epsilon(z, t)\|_X \\ & \leq \frac{(L_1 + L_2 (\|\omega\|_{C^{0,\gamma}(D)} + \|y\|_X + \|z\|_X))}{\epsilon^{2+\alpha(\gamma)}} \|y - z\|_X, \end{aligned}$$

where  $L_1, L_2$  are independent of  $\mathbf{u}, \mathbf{v}$  and depend upon peridynamic potential function  $f$  and influence function  $J$  and the exponent  $\alpha(\gamma)$  is given by

$$\alpha(\gamma) = \begin{cases} 0 & \text{if } \gamma \geq 1/2, \\ 1/2 - \gamma & \text{if } \gamma < 1/2. \end{cases}$$

Furthermore, for any  $y \in X$  and any  $t \in [0, T]$ , we have the bound

$$(12) \quad \|F^\epsilon(y, t)\|_X \leq \frac{L_3}{\epsilon^{2+\alpha(\gamma)}} (1 + \|\omega\|_{C^{0,\gamma}(D)} + \|y\|_X) + b,$$

where  $b = \sup_t \|\mathbf{b}(t)\|_{C^{0,\gamma}(D; \mathbb{R}^d)}$  and  $L_3$  is independent of  $y$ .

We easily see that on choosing  $z = 0$  in (11) that  $-\nabla PD^\epsilon(\mathbf{u})(\mathbf{x})$  is in  $C^{0,\gamma}(D; \mathbb{R}^3)$  provided that  $\mathbf{u}$  belongs to  $C^{0,\gamma}(D; \mathbb{R}^3)$ . Since  $-\nabla PD^\epsilon(\mathbf{u})(\mathbf{x})$  takes the value 0 on  $\partial D$ , we conclude that  $-\nabla PD^\epsilon(\mathbf{u})(\mathbf{x})$  belongs to  $C_0^{0,\gamma}(D; \mathbb{R}^3)$ .

In Theorem 6.1 of [23], the Lipschitz property of a peridynamic force is shown in  $X = L^2(D; \mathbb{R}^d) \times L^2(D; \mathbb{R}^d)$ . It is given by

$$(13) \quad \|F^\epsilon(y, t) - F^\epsilon(z, t)\|_X \leq \frac{L}{\epsilon^2} \|y - z\|_X \quad \forall y, z \in X, \forall t \in [0, T]$$

for all  $y, z \in L_0^2(D; \mathbb{R}^d)^2$ . For this case  $L$  does not depend on  $\mathbf{u}, \mathbf{v}$ . We now state the existence theorem.

The following theorem gives the existence and uniqueness of solution in any given time domain  $I_0 = (-T, T)$ .

**THEOREM 2** (existence and uniqueness of Hölder solutions of cohesive dynamics over finite time intervals). *For any initial condition  $x_0 \in X = C_0^{0,\gamma}(D; \mathbb{R}^d) \times C_0^{0,\gamma}(D; \mathbb{R}^d)$ ,*

time interval  $I_0 = (-T, T)$ , and right-hand side  $\mathbf{b}(t)$  continuous in time for  $t \in I_0$  such that  $\mathbf{b}(t)$  satisfies  $\sup_{t \in I_0} \|\mathbf{b}(t)\|_{C^{0,\gamma}} < \infty$ , there is a unique solution  $y(t) \in C^1(I_0; X)$  of

$$y(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau) d\tau,$$

or, equivalently,

$$y'(t) = F^\epsilon(y(t), t), \quad \text{with } y(0) = x_0,$$

where  $y(t)$  and  $y'(t)$  are Lipschitz continuous in time for  $t \in I_0$ .

The proof of this theorem is given in section 4. We now describe the finite difference scheme and analyze its convergence to Hölder continuous solutions of cohesive dynamics.

**3. Finite difference approximation.** In this section, we present the finite difference scheme and compute the rate of convergence. We first consider the semidiscrete approximation and prove the bound on energy of semidiscrete evolution in terms of initial energy and the work done by body forces.

Let  $h$  be the size of a mesh and let  $\Delta t$  be the size of the time step. We will keep  $\epsilon$  fixed and assume that  $h < \epsilon < 1$ . Let  $D_h = D \cap (h\mathbb{Z})^d$  be the discretization of material domain; see Figure 3. Let  $i \in \mathbb{Z}^d$  be the index such that  $\mathbf{x}_i = hi \in D$ . Let  $U_i$  be the unit cell of volume  $h^d$  corresponding to the grid point  $\mathbf{x}_i$ . The exact solution evaluated at grid points is denoted by  $(\mathbf{u}_i(t), \mathbf{v}_i(t))$ .

**3.1. Time discretization.** Let  $[0, T] \cap (\Delta t\mathbb{Z})$  be the discretization of time domain where  $\Delta t$  is the size of the time step. Denote the fully discrete solution at  $(t^k = k\Delta t, \mathbf{x}_i = ih)$  as  $(\hat{\mathbf{u}}_i^k, \hat{\mathbf{v}}_i^k)$ . Similarly, the exact solution evaluated at grid points is denoted by  $(\mathbf{u}_i^k, \mathbf{v}_i^k)$ . We enforce boundary condition  $\hat{\mathbf{u}}_i^k = \mathbf{0}$  for all  $\mathbf{x}_i \notin D$  and for all  $k$ .

We begin with the forward Euler time discretization, with respect to velocity, and the finite difference scheme for  $(\hat{\mathbf{u}}_i^k, \hat{\mathbf{v}}_i^k)$  is written as

$$(14) \quad \frac{\hat{\mathbf{u}}_i^{k+1} - \hat{\mathbf{u}}_i^k}{\Delta t} = \hat{\mathbf{v}}_i^{k+1},$$

$$(15) \quad \frac{\hat{\mathbf{v}}_i^{k+1} - \hat{\mathbf{v}}_i^k}{\Delta t} = -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \mathbf{b}_i^k.$$

The scheme is complemented with the discretized initial conditions  $\hat{\mathbf{u}}_i^0 = (\hat{\mathbf{u}}_0)_i$  and  $\hat{\mathbf{v}}_i^0 = (\hat{\mathbf{v}}_0)_i$ . If we substitute (14) into (15), we get a standard central difference scheme in time for a second order in time differential equation. Here we have assumed, without loss of generality,  $\rho = 1$ .

The piecewise constant extensions of the discrete sets  $\{\hat{\mathbf{u}}_i^k\}_{i \in \mathbb{Z}^d}$  and  $\{\hat{\mathbf{v}}_i^k\}_{i \in \mathbb{Z}^d}$  are given by

$$\hat{\mathbf{u}}^k(\mathbf{x}) := \sum_{i, \mathbf{x}_i \in D} \hat{\mathbf{u}}_i^k \chi_{U_i}(\mathbf{x}),$$

$$\hat{\mathbf{v}}^k(\mathbf{x}) := \sum_{i, \mathbf{x}_i \in D} \hat{\mathbf{v}}_i^k \chi_{U_i}(\mathbf{x}).$$

In this way we represent the finite difference solution as a piecewise constant function. We will show this function provides an  $L^2$  approximation of the exact solution.

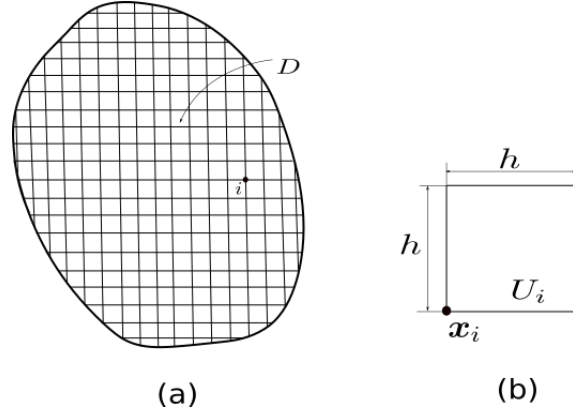


FIG. 3. (a) Typical mesh of size  $h$ . (b) Unit cell  $U_i$  corresponding to material point  $\mathbf{x}_i$ .

**3.1.1. Convergence results.** In this section we provide upper bounds on the rate of convergence of the discrete approximation to the solution of the peridynamic evolution. The  $L^2$  approximation error  $E^k$  at time  $t^k$ , for  $0 < t^k \leq T$ , is defined as

$$E^k := \left\| \hat{\mathbf{u}}^k - \mathbf{u}^k \right\|_{L^2(D; \mathbb{R}^d)} + \left\| \hat{\mathbf{v}}^k - \mathbf{v}^k \right\|_{L^2(D; \mathbb{R}^d)}.$$

The upper bound on the convergence rate of the approximation error is given by the following theorem.

**THEOREM 3** (convergence of finite difference approximation (forward Euler time discretization)). *Let  $\epsilon > 0$  be fixed. Let  $(\mathbf{u}, \mathbf{v})$  be the solution of peridynamic equation (10). We assume  $\mathbf{u}, \mathbf{v} \in C^2([0, T]; C_0^{0, \gamma}(D; \mathbb{R}^d))$ . Then the finite difference scheme given by (14) and (15) is consistent in both time and spatial discretization and converges to the exact solution uniformly in time with respect to the  $L^2(D; \mathbb{R}^d)$  norm. If we assume the error at the initial step is zero, then the error  $E^k$  at time  $t^k$  is bounded and to leading order in the time step  $\Delta t$  satisfies*

$$(16) \quad \sup_{0 \leq k \leq T/\Delta t} E^k \leq O \left( C_t \Delta t + C_s \frac{h^\gamma}{\epsilon^2} \right),$$

where constants  $C_s$  and  $C_t$  are independent of  $h$  and  $\Delta t$  and  $C_s$  depends on the Hölder norm of the solution and  $C_t$  depends on the  $L^2$  norms of time derivatives of the solution.

Here we have assumed the initial error to be zero for ease of exposition only.

We remark that the explicit constants leading to (16) can be large. The inequality that delivers (16) is given to leading order by

$$(17) \quad \sup_{0 \leq k \leq T/\Delta t} E^k \leq \exp [T(1 + 6\bar{C}/\epsilon^2)] T [C_t \Delta t + (C_s/\epsilon^2)h^\gamma],$$

where the constants  $\bar{C}$ ,  $C_t$ , and  $C_s$  are given by (35), (37), and (38). The explicit constant  $C_t$  depends on the spatial  $L^2$  norm of the time derivatives of the solution and  $C_s$  depends on the spatial Hölder continuity of the solution and the constant  $\bar{C}$ . This constant is bounded independently of horizon  $\epsilon$ . Although the constants are necessarily pessimistic they deliver a priori error estimates and an example is discussed in section 6.



An identical convergence rate can be established for the general one step scheme and we state it below.

**THEOREM 4** (convergence of finite difference approximation (general single step time discretization)). *Let us assume that the hypothesis of Theorem 3 holds. Fix  $\theta \in [0, 1]$ , and let  $(\hat{\mathbf{u}}^k, \hat{\mathbf{v}}^k)^T$  be the solution of the following finite difference equation:*

$$(18) \quad \frac{\hat{\mathbf{u}}_i^{k+1} - \hat{\mathbf{u}}_i^k}{\Delta t} = (1 - \theta)\hat{\mathbf{v}}_i^k + \theta\hat{\mathbf{v}}_i^{k+1},$$

$$(19) \quad \frac{\hat{\mathbf{v}}_i^{k+1} - \hat{\mathbf{v}}_i^k}{\Delta t} = (1 - \theta) \left( -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \mathbf{b}_i^k \right) + \theta \left( -\nabla PD^\epsilon(\hat{\mathbf{u}}^{k+1})(\mathbf{x}_i) + \mathbf{b}_i^{k+1} \right).$$

Then, for any fixed  $\theta \in [0, 1]$ , there exists a constant  $K > 0$  independent of  $(\hat{\mathbf{u}}^k, \hat{\mathbf{v}}^k)^T$  and  $(\mathbf{u}^k, \mathbf{v}^k)^T$ , such that for  $\Delta t < K\epsilon^2$  the finite difference scheme given by (18) and (19) is consistent in time and spatial discretization. If we assume the error at the initial step is zero, then the error  $E^k$  at time  $t^k$  is bounded and satisfies

$$\sup_{0 \leq k \leq T/\Delta t} E^k \leq O \left( C_t \Delta t + C_s \frac{h^\gamma}{\epsilon^2} \right).$$

The constant  $K$  is given by the explicit formula  $K = 1/\bar{C}$  where  $\bar{C}$  is described by (35). Furthermore for the Crank–Nicolson scheme,  $\theta = 1/2$ , if we assume the solutions  $\mathbf{u}, \mathbf{v}$  belong to  $C^3([0, T]; C_0^{0,\gamma}(D; \mathbb{R}^d))$ , then the approximation error  $E^k$  satisfies

$$\sup_{0 \leq k \leq T/\Delta t} E^k \leq O \left( \bar{C}_t (\Delta t)^2 + C_s \frac{h^\gamma}{\epsilon^2} \right),$$

where  $\bar{C}_t$  is independent of  $\Delta t$  and  $h$  and is given by (49).

As before we assume that the error in the initial data is zero for ease of exposition. The proofs of Theorems 3 and 4 are given in the following sections.

*Remark.* In Theorem 4, we have stated a condition on  $\Delta t$  for which the convergence estimate holds. This condition naturally occurs in the analysis and is related to the Lipschitz continuity of the peridynamic force with respect to the  $L^2$  norm; see (41).

**3.1.2. Error analysis.** Theorems 3 and 4 are proved along similar lines. In both cases we define the  $L^2$ -projections of the actual solutions onto the space of piecewise constant functions defined over the cells  $U_i$ . These are given as follows. Let  $(\tilde{\mathbf{u}}_i^k, \tilde{\mathbf{v}}_i^k)$  be the average of the exact solution  $(\mathbf{u}^k, \mathbf{v}^k)$  in the unit cell  $U_i$  given by

$$\begin{aligned} \tilde{\mathbf{u}}_i^k &:= \frac{1}{h^d} \int_{U_i} \mathbf{u}^k(\mathbf{x}) d\mathbf{x}, \\ \tilde{\mathbf{v}}_i^k &:= \frac{1}{h^d} \int_{U_i} \mathbf{v}^k(\mathbf{x}) d\mathbf{x}, \end{aligned}$$

and the  $L^2$  projection of the solution onto piecewise constant functions are  $(\tilde{\mathbf{u}}^k, \tilde{\mathbf{v}}^k)$  given by

$$(20) \quad \tilde{\mathbf{u}}^k(\mathbf{x}) := \sum_{i, \mathbf{x}_i \in D} \tilde{\mathbf{u}}_i^k \chi_{U_i}(\mathbf{x}),$$

$$(21) \quad \tilde{\mathbf{v}}^k(\mathbf{x}) := \sum_{i, \mathbf{x}_i \in D} \tilde{\mathbf{v}}_i^k \chi_{U_i}(\mathbf{x}).$$

The error between  $(\hat{\mathbf{u}}^k, \hat{\mathbf{v}}^k)^T$  with  $(\mathbf{u}(t^k), \mathbf{v}(t^k))^T$  is now split into two parts. From the triangle inequality, we have

$$\begin{aligned} \|\hat{\mathbf{u}}^k - \mathbf{u}(t^k)\|_{L^2(D; \mathbb{R}^d)} &\leq \|\hat{\mathbf{u}}^k - \tilde{\mathbf{u}}^k\|_{L^2(D; \mathbb{R}^d)} + \|\tilde{\mathbf{u}}^k - \mathbf{u}^k\|_{L^2(D; \mathbb{R}^d)}, \\ \|\hat{\mathbf{v}}^k - \mathbf{v}(t^k)\|_{L^2(D; \mathbb{R}^d)} &\leq \|\hat{\mathbf{v}}^k - \tilde{\mathbf{v}}^k\|_{L^2(D; \mathbb{R}^d)} + \|\tilde{\mathbf{v}}^k - \mathbf{v}^k\|_{L^2(D; \mathbb{R}^d)}. \end{aligned}$$

In subsection 3.1.3 and subsection 3.2 we will show that the error between the  $L^2$  projections of the actual solution and the discrete approximation for both forward Euler and implicit one step methods decay according to

$$(22) \quad \sup_{0 \leq k \leq T/\Delta t} \left( \|\hat{\mathbf{u}}^k - \tilde{\mathbf{u}}^k\|_{L^2(D; \mathbb{R}^d)} + \|\hat{\mathbf{v}}^k - \tilde{\mathbf{v}}^k\|_{L^2(D; \mathbb{R}^d)} \right) = O\left(\Delta t + \frac{h^\gamma}{\epsilon^2}\right).$$

In what follows we can estimate the terms

$$(23) \quad \|\tilde{\mathbf{u}}^k - \mathbf{u}(t^k)\|_{L^2} \quad \text{and} \quad \|\tilde{\mathbf{v}}^k - \mathbf{v}(t^k)\|_{L^2}$$

and show they go to zero at a rate of  $h^\gamma$  uniformly in time. The estimates given by (22) together with the  $O(h^\gamma)$  estimates for (23) establish Theorems 3 and 4. We now establish the  $L^2$  estimates for the differences  $\tilde{\mathbf{u}}^k - \mathbf{u}(t^k)$  and  $\tilde{\mathbf{v}}^k - \mathbf{v}(t^k)$ .

We write

$$\begin{aligned} &\|\tilde{\mathbf{u}}^k - \mathbf{u}^k\|_{L^2(D; \mathbb{R}^d)}^2 \\ &= \sum_{i, \mathbf{x}_i \in D} \int_{U_i} \left| \tilde{\mathbf{u}}^k(\mathbf{x}) - \mathbf{u}^k(\mathbf{x}) \right|^2 d\mathbf{x} \\ &= \sum_{i, \mathbf{x}_i \in D} \int_{U_i} \left| \frac{1}{h^d} \int_{U_i} (\mathbf{u}^k(\mathbf{y}) - \mathbf{u}^k(\mathbf{x})) d\mathbf{y} \right|^2 d\mathbf{x} \\ &= \sum_{i, \mathbf{x}_i \in D} \int_{U_i} \left[ \frac{1}{h^{2d}} \int_{U_i} \int_{U_i} (\mathbf{u}^k(\mathbf{y}) - \mathbf{u}^k(\mathbf{x})) \cdot (\mathbf{u}^k(\mathbf{z}) - \mathbf{u}^k(\mathbf{x})) d\mathbf{y} d\mathbf{z} \right] d\mathbf{x} \\ (24) \quad &\leq \sum_{i, \mathbf{x}_i \in D} \int_{U_i} \left[ \frac{1}{h^d} \int_{U_i} |\mathbf{u}^k(\mathbf{y}) - \mathbf{u}^k(\mathbf{x})|^2 d\mathbf{y} \right] d\mathbf{x}, \end{aligned}$$

where we used Cauchy's inequality and Jensen's inequality. For  $\mathbf{x}, \mathbf{y} \in U_i$ ,  $|\mathbf{x} - \mathbf{y}| \leq ch$ , where  $c = \sqrt{2}$  for  $d = 2$  and  $c = \sqrt{3}$  for  $d = 3$ . Since  $\mathbf{u} \in C_0^{0, \gamma}$  we have

$$\begin{aligned} |\mathbf{u}^k(\mathbf{x}) - \mathbf{u}^k(\mathbf{y})| &= |\mathbf{x} - \mathbf{y}|^\gamma \frac{|\mathbf{u}^k(\mathbf{y}) - \mathbf{u}^k(\mathbf{x})|}{|\mathbf{x} - \mathbf{y}|^\gamma} \\ (25) \quad &\leq c^\gamma h^\gamma \|\mathbf{u}^k\|_{C^{0, \gamma}(D; \mathbb{R}^d)} \leq c^\gamma h^\gamma \sup_t \|\mathbf{u}(t)\|_{C^{0, \gamma}(D; \mathbb{R}^d)} \end{aligned}$$

and substitution in (24) gives

$$\begin{aligned} \|\tilde{\mathbf{u}}^k - \mathbf{u}^k\|_{L^2(D; \mathbb{R}^d)}^2 &\leq c^{2\gamma} h^{2\gamma} \sum_{i, \mathbf{x}_i \in D} \int_{U_i} d\mathbf{x} \left( \sup_t \|\mathbf{u}(t)\|_{C^{0, \gamma}(D; \mathbb{R}^d)} \right)^2 \\ &\leq c^{2\gamma} |D| h^{2\gamma} \left( \sup_t \|\mathbf{u}(t)\|_{C^{0, \gamma}(D; \mathbb{R}^d)} \right)^2. \end{aligned}$$

A similar estimate can be derived for  $\|\tilde{\mathbf{v}}^k - \mathbf{v}^k\|_{L^2}$  and substitution of the estimates into (23) gives

$$\sup_k \left( \left\| \tilde{\mathbf{u}}^k - \mathbf{u}(t^k) \right\|_{L^2(D; \mathbb{R}^d)} + \left\| \tilde{\mathbf{v}}^k - \mathbf{v}(t^k) \right\|_{L^2(D; \mathbb{R}^d)} \right) = O(h^\gamma).$$

In the next section we establish the error estimate (22) for both forward Euler and general one step schemes in subsection 3.1.3 and subsection 3.2.

**3.1.3. Error analysis for approximation of  $L^2$  projection of the exact solution.** In this subsection, we estimate the difference between approximate solution  $(\hat{\mathbf{u}}^k, \hat{\mathbf{v}}^k)$  and the  $L^2$  projection of the exact solution onto piecewise constant functions given by  $(\tilde{\mathbf{u}}^k, \tilde{\mathbf{v}}^k)$ ; see (20) and (21). Let the differences be denoted by  $\mathbf{e}^k(u) := \hat{\mathbf{u}}^k - \tilde{\mathbf{u}}^k$  and  $\mathbf{e}^k(v) := \hat{\mathbf{v}}^k - \tilde{\mathbf{v}}^k$ , and their evaluation at grid points are  $\mathbf{e}_i^k(u) := \hat{\mathbf{u}}_i^k - \tilde{\mathbf{u}}_i^k$  and  $\mathbf{e}_i^k(v) := \hat{\mathbf{v}}_i^k - \tilde{\mathbf{v}}_i^k$ . Subtracting  $(\tilde{\mathbf{u}}_i^{k+1} - \tilde{\mathbf{u}}_i^k)/\Delta t$  from (14) gives

$$\begin{aligned} & \frac{\hat{\mathbf{u}}_i^{k+1} - \hat{\mathbf{u}}_i^k}{\Delta t} - \frac{\tilde{\mathbf{u}}_i^{k+1} - \tilde{\mathbf{u}}_i^k}{\Delta t} \\ &= \hat{\mathbf{v}}_i^{k+1} - \frac{\tilde{\mathbf{u}}_i^{k+1} - \tilde{\mathbf{u}}_i^k}{\Delta t} \\ &= \hat{\mathbf{v}}_i^{k+1} - \tilde{\mathbf{v}}_i^{k+1} + \left( \tilde{\mathbf{v}}_i^{k+1} - \frac{\partial \tilde{\mathbf{u}}_i^{k+1}}{\partial t} \right) + \left( \frac{\partial \tilde{\mathbf{u}}_i^{k+1}}{\partial t} - \frac{\tilde{\mathbf{u}}_i^{k+1} - \tilde{\mathbf{u}}_i^k}{\Delta t} \right). \end{aligned}$$

Taking the average over unit cell  $U_i$  of the exact peridynamic equation (10) at time  $t^k$ , we will get

$$\tilde{\mathbf{v}}_i^{k+1} - \frac{\partial \tilde{\mathbf{u}}_i^{k+1}}{\partial t} = 0.$$

Therefore, the equation for  $\mathbf{e}_i^k(u)$  is given by

$$(26) \quad \mathbf{e}_i^{k+1}(u) = \mathbf{e}_i^k(u) + \Delta t \mathbf{e}_i^{k+1}(v) + \Delta t \tau_i^k(u),$$

where we identify the discretization error as

$$(27) \quad \tau_i^k(u) := \frac{\partial \tilde{\mathbf{u}}_i^{k+1}}{\partial t} - \frac{\tilde{\mathbf{u}}_i^{k+1} - \tilde{\mathbf{u}}_i^k}{\Delta t}.$$

Similarly, we subtract  $(\tilde{\mathbf{v}}_i^{k+1} - \tilde{\mathbf{v}}_i^k)/\Delta t$  from (15) and add and subtract terms to get

$$\begin{aligned} & \frac{\hat{\mathbf{v}}_i^{k+1} - \hat{\mathbf{v}}_i^k}{\Delta t} - \frac{\tilde{\mathbf{v}}_i^{k+1} - \tilde{\mathbf{v}}_i^k}{\Delta t} = -\nabla P D^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \mathbf{b}_i^k - \frac{\partial \mathbf{v}_i^k}{\partial t} + \left( \frac{\partial \mathbf{v}_i^k}{\partial t} - \frac{\tilde{\mathbf{v}}_i^{k+1} - \tilde{\mathbf{v}}_i^k}{\Delta t} \right) \\ &= -\nabla P D^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \mathbf{b}_i^k - \frac{\partial \mathbf{v}_i^k}{\partial t} \\ &+ \left( \frac{\partial \tilde{\mathbf{v}}_i^k}{\partial t} - \frac{\tilde{\mathbf{v}}_i^{k+1} - \tilde{\mathbf{v}}_i^k}{\Delta t} \right) + \left( \frac{\partial \mathbf{v}_i^k}{\partial t} - \frac{\partial \tilde{\mathbf{v}}_i^k}{\partial t} \right), \end{aligned} \tag{28}$$

where we identify  $\tau_i^k(v)$  as follows:

$$(29) \quad \tau_i^k(v) := \frac{\partial \tilde{\mathbf{v}}_i^k}{\partial t} - \frac{\tilde{\mathbf{v}}_i^{k+1} - \tilde{\mathbf{v}}_i^k}{\Delta t}.$$

Note that in  $\tau^k(u)$  we have

$$\frac{\partial \tilde{\mathbf{u}}_i^{k+1}}{\partial t}$$

and from the exact peridynamic equation, we have

$$(30) \quad \mathbf{b}_i^k - \frac{\partial \mathbf{v}_i^k}{\partial t} = \nabla PD^\epsilon(\mathbf{u}^k)(\mathbf{x}_i).$$

Combining (28), (29), and (30), we get

$$\begin{aligned} \mathbf{e}_i^{k+1}(v) &= \mathbf{e}_i^k(v) + \Delta t \tau_i^k(v) + \Delta t \left( \frac{\partial \mathbf{v}_i^k}{\partial t} - \frac{\partial \tilde{\mathbf{v}}_i^k}{\partial t} \right) \\ &\quad + \Delta t \left( -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\mathbf{u}^k)(\mathbf{x}_i) \right) \\ &= \mathbf{e}_i^k(v) + \Delta t \tau_i^k(v) + \Delta t \left( \frac{\partial \mathbf{v}_i^k}{\partial t} - \frac{\partial \tilde{\mathbf{v}}_i^k}{\partial t} \right) \\ &\quad + \Delta t \left( -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) \right) \\ &\quad + \Delta t \left( -\nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\mathbf{u}^k)(\mathbf{x}_i) \right). \end{aligned}$$

The spatial discretization errors  $\sigma_i^k(u)$  and  $\sigma_i^k(v)$  are given by

$$(31) \quad \sigma_i^k(u) := \left( -\nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\mathbf{u}^k)(\mathbf{x}_i) \right),$$

$$(32) \quad \sigma_i^k(v) := \frac{\partial \mathbf{v}_i^k}{\partial t} - \frac{\partial \tilde{\mathbf{v}}_i^k}{\partial t}.$$

We finally have

$$(33) \quad \begin{aligned} \mathbf{e}_i^{k+1}(v) &= \mathbf{e}_i^k(v) + \Delta t \left( \tau_i^k(v) + \sigma_i^k(u) + \sigma_i^k(v) \right) \\ &\quad + \Delta t \left( -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) \right). \end{aligned}$$

We now show the consistency and stability properties of the numerical scheme.

**3.1.4. Consistency.** We deal with the error in time discretization and the error in spatial discretization error separately. The time discretization error follows easily using the Taylor series while the spatial discretization error uses properties of the nonlinear peridynamic force.

**Time discretization.** We first estimate the time discretization error. A Taylor series expansion is used to estimate  $\tau_i^k(u)$  as follows:

$$\begin{aligned} \tau_i^k(u) &= \frac{1}{h^d} \int_{U_i} \left( \frac{\partial \mathbf{u}^k(\mathbf{x})}{\partial t} - \frac{\mathbf{u}^{k+1}(\mathbf{x}) - \mathbf{u}^k(\mathbf{x})}{\Delta t} \right) d\mathbf{x}, \\ &= \frac{1}{h^d} \int_{U_i} \left( -\frac{1}{2} \frac{\partial^2 \mathbf{u}^k(\mathbf{x})}{\partial t^2} \Delta t + O((\Delta t)^2) \right) d\mathbf{x}. \end{aligned}$$

Computing the  $L^2$  norm of  $\tau_i^k(u)$  and using Jensen's inequality gives

$$\begin{aligned} \|\tau_i^k(u)\|_{L^2(D; \mathbb{R}^d)} &\leq \frac{\Delta t}{2} \left\| \frac{\partial^2 \mathbf{u}^k}{\partial t^2} \right\|_{L^2(D; \mathbb{R}^d)} + O((\Delta t)^2) \\ &\leq \frac{\Delta t}{2} \sup_t \left\| \frac{\partial^2 \mathbf{u}(t)}{\partial t^2} \right\|_{L^2(D; \mathbb{R}^d)} + O((\Delta t)^2). \end{aligned}$$

Similarly, we have

$$\|\tau^k(v)\|_{L^2(D;\mathbb{R}^d)} = \frac{\Delta t}{2} \sup_t \left\| \frac{\partial^2 \mathbf{v}(t)}{\partial t^2} \right\|_{L^2(D;\mathbb{R}^d)} + O((\Delta t)^2).$$

**Spatial discretization.** We now estimate the spatial discretization error. Substituting the definition of  $\tilde{\mathbf{v}}^k$  and following the similar steps employed in (25) gives

$$\begin{aligned} |\sigma_i^k(v)| &= \left| \frac{\partial \mathbf{v}_i^k}{\partial t} - \frac{1}{h^d} \int_{U_i} \frac{\partial \mathbf{v}^k(\mathbf{x})}{\partial t} d\mathbf{x} \right| \leq c^\gamma h^\gamma \int_{U_i} \frac{1}{|\mathbf{x}_i - \mathbf{x}|^\gamma} \left| \frac{\partial \mathbf{v}^k(\mathbf{x}_i)}{\partial t} - \frac{\partial \mathbf{v}^k(\mathbf{x})}{\partial t} \right| d\mathbf{x} \\ &\leq c^\gamma h^\gamma \left\| \frac{\partial \mathbf{v}^k}{\partial t} \right\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \leq c^\gamma h^\gamma \sup_t \left\| \frac{\partial \mathbf{v}(t)}{\partial t} \right\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \end{aligned}$$

Taking the  $L^2$  norm of error  $\sigma_i^k(v)$  and substituting the estimate above delivers

$$\|\sigma^k(v)\|_{L^2(D;\mathbb{R}^d)} \leq h^\gamma c^\gamma \sqrt{|D|} \sup_t \left\| \frac{\partial \mathbf{v}(t)}{\partial t} \right\|_{C^{0,\gamma}(D;\mathbb{R}^d)}.$$

Now we estimate  $|\sigma_i^k(u)|$ . We use the notation  $\bar{\mathbf{u}}^k(\mathbf{x}) := \mathbf{u}^k(\mathbf{x} + \epsilon \boldsymbol{\xi}) - \mathbf{u}^k(\mathbf{x})$  and  $\tilde{\bar{\mathbf{u}}}^k(\mathbf{x}) := \tilde{\mathbf{u}}(\mathbf{x} + \epsilon \boldsymbol{\xi}) - \tilde{\mathbf{u}}^k(\mathbf{x})$  and choose  $\mathbf{u} = \mathbf{u}^k$  and  $\mathbf{v} = \tilde{\mathbf{u}}^k$  in (76) to find that

$$\begin{aligned} |\sigma_i^k(u)| &= \left| -\nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\mathbf{u}^k)(\mathbf{x}_i) \right| \\ (34) \quad &\leq \frac{2C_2}{\epsilon \omega_d} \left| \int_{H_1(\mathbf{0})} J(|\boldsymbol{\xi}|) \frac{\left| \mathbf{u}^k(\mathbf{x}_i + \epsilon \boldsymbol{\xi}) - \tilde{\mathbf{u}}^k(\mathbf{x}_i + \epsilon \boldsymbol{\xi}) - (\mathbf{u}^k(\mathbf{x}_i) - \tilde{\mathbf{u}}^k(\mathbf{x}_i)) \right|}{\epsilon |\boldsymbol{\xi}|} d\boldsymbol{\xi} \right|. \end{aligned}$$

Here  $C_2$  is the maximum of the second derivative of the profile describing the potential given by (71). Following the earlier analysis (see (25)), we find that

$$\begin{aligned} \left| \mathbf{u}^k(\mathbf{x}_i + \epsilon \boldsymbol{\xi}) - \tilde{\mathbf{u}}^k(\mathbf{x}_i + \epsilon \boldsymbol{\xi}) \right| &\leq c^\gamma h^\gamma \sup_t \|\mathbf{u}(t)\|_{C^{0,\gamma}(D;\mathbb{R}^d)}, \\ \left| \mathbf{u}^k(\mathbf{x}_i) - \tilde{\mathbf{u}}^k(\mathbf{x}_i) \right| &\leq c^\gamma h^\gamma \sup_t \|\mathbf{u}(t)\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \end{aligned}$$

For reference, we define the constant

$$(35) \quad \bar{C} = \frac{C_2}{\omega_d} \int_{H_1(\mathbf{0})} J(|\boldsymbol{\xi}|) \frac{1}{|\boldsymbol{\xi}|} d\boldsymbol{\xi}.$$

We now focus on (34). We substitute the above two inequalities to get

$$\begin{aligned} |\sigma_i^k(u)| &\leq \frac{2C_2}{\epsilon^2 \omega_d} \left| \int_{H_1(\mathbf{0})} J(|\boldsymbol{\xi}|) \frac{1}{|\boldsymbol{\xi}|} \right. \\ &\quad \left. \left( \left| \mathbf{u}^k(\mathbf{x}_i + \epsilon \boldsymbol{\xi}) - \tilde{\mathbf{u}}^k(\mathbf{x}_i + \epsilon \boldsymbol{\xi}) \right| + \left| \mathbf{u}^k(\mathbf{x}_i) - \tilde{\mathbf{u}}^k(\mathbf{x}_i) \right| \right) d\boldsymbol{\xi} \right| \\ &\leq 4h^\gamma c^\gamma \frac{\bar{C}}{\epsilon^2} \sup_t \|\mathbf{u}(t)\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \end{aligned}$$

Therefore, we have

$$\|\sigma^k(u)\|_{L^2(D;\mathbb{R}^d)} \leq h^\gamma \left( 4c^\gamma \sqrt{|D|} \frac{\bar{C}}{\epsilon^2} \sup_t \|\mathbf{u}(t)\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \right).$$

This completes the proof of consistency of numerical approximation.

**3.1.5. Stability.** Let  $e^k$  be the total error at the  $k$ th time step. It is defined as

$$e^k := \|e^k(u)\|_{L^2(D;\mathbb{R}^d)} + \|e^k(v)\|_{L^2(D;\mathbb{R}^d)}.$$

To simplify the calculations, we define the new term  $\tau$  as

$$\begin{aligned} \tau := & \sup_t \left( \|\tau^k(u)\|_{L^2(D;\mathbb{R}^d)} + \|\tau^k(v)\|_{L^2(D;\mathbb{R}^d)} \right. \\ & \left. + \|\sigma^k(u)\|_{L^2(D;\mathbb{R}^d)} + \|\sigma^k(v)\|_{L^2(D;\mathbb{R}^d)} \right). \end{aligned}$$

From our consistency analysis, we know that to leading order

$$(36) \quad \tau \leq C_t \Delta t + \frac{C_s}{\epsilon^2} h^\gamma,$$

where

$$(37) \quad C_t := \frac{1}{2} \sup_t \left\| \frac{\partial^2 \mathbf{u}(t)}{\partial t^2} \right\|_{L^2(D;\mathbb{R}^d)} + \frac{1}{2} \sup_t \left\| \frac{\partial^3 \mathbf{u}(t)}{\partial t^3} \right\|_{L^2(D;\mathbb{R}^d)},$$

$$(38) \quad C_s := c^\gamma \sqrt{|D|} \left[ \epsilon^2 \sup_t \left\| \frac{\partial^2 \mathbf{u}(t)}{\partial t^2} \right\|_{C^{0,\gamma}(D;\mathbb{R}^d)} + 4\bar{C} \sup_t \|\mathbf{u}(t)\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \right].$$

We take the  $L^2$  norms of (26) and (33) and add them. Noting the definition of  $\tau$  as above, we get

$$(39) \quad \begin{aligned} e^{k+1} \leq & e^k + \Delta t \|e^{k+1}(v)\|_{L^2(D;\mathbb{R}^d)} + \Delta t \tau \\ & + \Delta t \left( \sum_i h^d \left| -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) \right|^2 \right)^{1/2}. \end{aligned}$$

We need only estimate the last term in above equation. Similar to (34), we have

$$\begin{aligned} & \left| -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) \right| \\ & \leq \frac{2C_2}{\epsilon^2 \omega_d} \left| \int_{H_1(\mathbf{0})} J(|\xi|) \frac{1}{|\xi|} \left| \hat{\mathbf{u}}^k(\mathbf{x}_i + \epsilon \xi) - \tilde{\mathbf{u}}^k(\mathbf{x}_i + \epsilon \xi) - (\hat{\mathbf{u}}^k(\mathbf{x}_i) - \tilde{\mathbf{u}}^k(\mathbf{x}_i)) \right| d\xi \right| \\ & = \frac{2C_2}{\epsilon^2 \omega_d} \left| \int_{H_1(\mathbf{0})} J(|\xi|) \frac{1}{|\xi|} \left| e^k(u)(\mathbf{x}_i + \epsilon \xi) - e^k(u)(\mathbf{x}_i) \right| d\xi \right| \\ & \leq \frac{2C_2}{\epsilon^2 \omega_d} \left| \int_{H_1(\mathbf{0})} J(|\xi|) \frac{1}{|\xi|} (|e^k(u)(\mathbf{x}_i + \epsilon \xi)| + |e^k(u)(\mathbf{x}_i)|) d\xi \right|. \end{aligned}$$

By  $e^k(u)(\mathbf{x})$  we mean evaluation of piecewise extension of set  $\{e_i^k(u)\}_i$  at  $\mathbf{x}$ . We proceed further as follows:

$$\begin{aligned} & \left| -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) \right|^2 \\ & \leq \left( \frac{2C_2}{\epsilon^2 \omega_d} \right)^2 \int_{H_1(\mathbf{0})} \int_{H_1(\mathbf{0})} J(|\xi|) J(|\eta|) \frac{1}{|\xi|} \frac{1}{|\eta|} \\ & \quad (|e^k(u)(\mathbf{x}_i + \epsilon \xi)| + |e^k(u)(\mathbf{x}_i)|) (|e^k(u)(\mathbf{x}_i + \epsilon \eta)| + |e^k(u)(\mathbf{x}_i)|) d\xi d\eta. \end{aligned}$$

Using inequality  $|ab| \leq (|a|^2 + |b|^2)/2$ , we get

$$\begin{aligned} & (|e^k(u)(\mathbf{x}_i + \epsilon\xi)| + |e^k(u)(\mathbf{x}_i)|) (|e^k(u)(\mathbf{x}_i + \epsilon\eta)| + |e^k(u)(\mathbf{x}_i)|) \\ & \leq 3 \left( |e^k(u)(\mathbf{x}_i + \epsilon\xi)|^2 + |e^k(u)(\mathbf{x}_i + \epsilon\eta)|^2 + |e^k(u)(\mathbf{x}_i)|^2 \right) \end{aligned}$$

and

$$\begin{aligned} & \sum_{\mathbf{x}_i \in D} h^d \left| -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) \right|^2 \\ & \leq \left( \frac{2C_2}{\epsilon^2 \omega_d} \right)^2 \int_{H_1(\mathbf{0})} \int_{H_1(\mathbf{0})} J(|\xi|)J(|\eta|) \frac{1}{|\xi|} \frac{1}{|\eta|} \\ & \quad \sum_{\mathbf{x}_i \in D} h^d 3 \left( |e^k(u)(\mathbf{x}_i + \epsilon\xi)|^2 + |e^k(u)(\mathbf{x}_i + \epsilon\eta)|^2 + |e^k(u)(\mathbf{x}_i)|^2 \right) d\xi d\eta. \end{aligned}$$

Since  $e^k(u)(\mathbf{x}) = \sum_{\mathbf{x}_i \in D} e_i^k(u)\chi_{U_i}(\mathbf{x})$ , we have

$$(40) \quad \sum_{\mathbf{x}_i \in D} h^d \left| -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) \right|^2 \leq \frac{(6\bar{C})^2}{\epsilon^4} \|e^k(u)\|_{L^2(D; \mathbb{R}^d)}^2,$$

where  $\bar{C}$  is given by (35). In summary, (40) shows the Lipschitz continuity of the peridynamic force with respect to the  $L^2$  norm; see (13), expressed in this context as

$$(41) \quad \|\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}) - \nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x})\|_{L^2(D; \mathbb{R}^d)} \leq \frac{(6\bar{C})}{\epsilon^2} \|e^k(u)\|_{L^2(D; \mathbb{R}^d)}.$$

Finally, we substitute the above inequality in (39) to get

$$e^{k+1} \leq e^k + \Delta t \|e^{k+1}(v)\|_{L^2(D; \mathbb{R}^d)} + \Delta t \tau + \Delta t \frac{6\bar{C}}{\epsilon^2} \|e^k(u)\|_{L^2(D; \mathbb{R}^d)}.$$

We add the positive quantity  $\Delta t \|e^{k+1}(u)\|_{L^2(D; \mathbb{R}^d)} + \Delta t 6\bar{C}/\epsilon^2 \|e^k(v)\|_{L^2(D; \mathbb{R}^d)}$  to the right-hand side of the above equation to get

$$\begin{aligned} e^{k+1} & \leq (1 + \Delta t 6\bar{C}/\epsilon^2) e^k + \Delta t e^{k+1} + \Delta t \tau \\ \Rightarrow e^{k+1} & \leq \frac{(1 + \Delta t 6\bar{C}/\epsilon^2)}{1 - \Delta t} e^k + \frac{\Delta t}{1 - \Delta t} \tau. \end{aligned}$$

We recursively substitute  $e^j$  on the above as follows:

$$\begin{aligned} (42) \quad e^{k+1} & \leq \frac{(1 + \Delta t 6\bar{C}/\epsilon^2)}{1 - \Delta t} e^k + \frac{\Delta t}{1 - \Delta t} \tau \\ & \leq \left( \frac{(1 + \Delta t 6\bar{C}/\epsilon^2)}{1 - \Delta t} \right)^2 e^{k-1} + \frac{\Delta t}{1 - \Delta t} \tau \left( 1 + \frac{(1 + \Delta t 6\bar{C}/\epsilon^2)}{1 - \Delta t} \right) \\ & \leq \dots \\ & \leq \left( \frac{(1 + \Delta t 6\bar{C}/\epsilon^2)}{1 - \Delta t} \right)^{k+1} e^0 + \frac{\Delta t}{1 - \Delta t} \tau \sum_{j=0}^k \left( \frac{(1 + \Delta t 6\bar{C}/\epsilon^2)}{1 - \Delta t} \right)^{k-j}. \end{aligned}$$

Since  $1/(1 - \Delta t) = 1 + \Delta t + \Delta t^2 + O(\Delta t^3)$ , we have

$$\frac{(1 + \Delta t 6\bar{C}/\epsilon^2)}{1 - \Delta t} \leq 1 + (1 + 6\bar{C}/\epsilon^2)\Delta t + (1 + 6\bar{C}/\epsilon^2)\Delta t^2 + O(\bar{C}/\epsilon^2)O(\Delta t^3).$$

Now, for any  $k \leq T/\Delta t$ , using identity  $(1 + a)^k \leq \exp[ka]$  for  $a \leq 0$ , we have

$$\begin{aligned} & \left( \frac{1 + \Delta t 6\bar{C}/\epsilon^2}{1 - \Delta t} \right)^k \\ & \leq \exp [k(1 + 6\bar{C}/\epsilon^2)\Delta t + k(1 + 6\bar{C}/\epsilon^2)\Delta t^2 + kO(\bar{C}/\epsilon^2)O(\Delta t^3)] \\ & \leq \exp [T(1 + 6\bar{C}/\epsilon^2) + T(1 + 6\bar{C}/\epsilon^2)\Delta t + O(T\bar{C}/\epsilon^2)O(\Delta t^2)]. \end{aligned}$$

We write the above equation in more compact form as follows:

$$\begin{aligned} & \left( \frac{1 + \Delta t 6\bar{C}/\epsilon^2}{1 - \Delta t} \right)^k \\ & \leq \exp [T(1 + 6\bar{C}/\epsilon^2)(1 + \Delta t + O(\Delta t^2))]. \end{aligned}$$

We use the above estimate in (42) and get the following inequality for  $e^k$ :

$$\begin{aligned} e^{k+1} & \leq \exp [T(1 + 6\bar{C}/\epsilon^2)(1 + \Delta t + O(\Delta t^2))] (e^0 + (k + 1)\tau\Delta t/(1 - \Delta t)) \\ & \leq \exp [T(1 + 6\bar{C}/\epsilon^2)(1 + \Delta t + O(\Delta t^2))] (e^0 + T\tau(1 + \Delta t + O(\Delta t^2))), \end{aligned}$$

where we used the fact that  $1/(1 - \Delta t) = 1 + \Delta t + O(\Delta t^2)$ .

Assuming the error in initial data is zero, i.e.,  $e^0 = 0$ , and noting the estimate of  $\tau$  in (36), we have

$$\sup_k e^k \leq \exp [T(1 + 6\bar{C}/\epsilon^2)] T\tau,$$

and we conclude to leading order that

$$(43) \quad \sup_k e^k \leq \exp [T(1 + 6\bar{C}/\epsilon^2)] T [C_t\Delta t + (C_s/\epsilon^2)h^\gamma].$$

Here the constants  $C_t$  and  $C_s$  are given by (37) and (38). This shows the stability of the numerical scheme. We now address the general one step time discretization.

**3.2. Extension to the implicit schemes.** Let  $\theta \in [0, 1]$  be the parameter which controls the contribution of the implicit and explicit schemes. Let  $(\hat{\mathbf{u}}^k, \hat{\mathbf{v}}^k)$  be the solution of (18) and (19) for given fixed  $\theta$ .

The forward Euler scheme, backward Euler scheme, and Crank–Nicolson scheme correspond to the choices  $\theta = 0$ ,  $\theta = 1$ , and  $\theta = 1/2$ , respectively.

To simplify the equations, we define  $\Theta$  acting on discrete set  $\{f^k\}_k$  as  $\Theta f^k := (1 - \theta)f^k + \theta f^{k+1}$ . By  $\Theta \|f^k\|$ , we mean  $(1 - \theta)\|f^k\| + \theta\|f^{k+1}\|$ . Following the same steps as in the case of forward Euler, we write down the equation for  $\mathbf{e}_i^k(u) := \hat{\mathbf{u}}_i^k - \tilde{\mathbf{u}}_i^k$  and  $\mathbf{e}_i^k(v) := \hat{\mathbf{v}}_i^k - \tilde{\mathbf{v}}_i^k$  as follows:

$$\begin{aligned} (44) \quad \mathbf{e}_i^{k+1}(u) & = \mathbf{e}_i^k(u) + \Delta t \Theta \mathbf{e}_i^k(v) + \Delta t \Theta \tau_i^k(u), \\ \mathbf{e}_i^{k+1}(v) & = \mathbf{e}_i^k(v) + \Delta t \Theta \sigma_i^k(u) + \Delta t \Theta \sigma_i^k(v) + \Delta t \Theta \tau_i^k(v) \\ & \quad + \Delta t(1 - \theta) \left( -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) \right) \\ (45) \quad & \quad + \Delta t\theta \left( -\nabla PD^\epsilon(\hat{\mathbf{u}}^{k+1})(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^{k+1})(\mathbf{x}_i) \right), \end{aligned}$$

where  $\tau_i^k(v), \sigma_i^k(u), \sigma_i^k(v)$  are defined in (29), (31), and (32), respectively. In this section  $\tau^k(u)$  is defined as follows:

$$\tau_i^k(u) := \frac{\partial \tilde{\mathbf{u}}_i^k}{\partial t} - \frac{\tilde{\mathbf{u}}_i^{k+1} - \tilde{\mathbf{u}}_i^k}{\Delta t}.$$



We take the  $L^2$  norm of  $\mathbf{e}^k(u)(\mathbf{x})$  and  $\mathbf{e}^k(v)(\mathbf{x})$ , and for brevity we denote the  $L^2$  norm by  $\|\cdot\|$ . Recall that  $\mathbf{e}^k(u)$  and  $\mathbf{e}^k(v)$  are the piecewise constant extensions of  $\{\mathbf{e}_i^k(u)\}_i$  and  $\{\mathbf{e}_i^k(v)\}_i$ , and we get

(46)

$$\begin{aligned} \|\mathbf{e}^{k+1}(u)\| &\leq \|\mathbf{e}^k(u)\| + \Delta t \Theta \|\mathbf{e}^k(v)\| + \Delta t \Theta \|\tau^k(u)\|, \\ \|\mathbf{e}^{k+1}(v)\| &\leq \|\mathbf{e}^k(v)\| + \Delta t (\Theta \|\sigma^k(u)\| + \Theta \|\sigma^k(v)\| + \Theta \|\tau^k(v)\|) \\ &\quad + \Delta t (1 - \theta) \left( \sum_{i, \mathbf{x}_i \in D} h^d \left| -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) \right|^2 \right)^{1/2} \\ (47) \quad &\quad + \Delta t \theta \left( \sum_{i, \mathbf{x}_i \in D} h^d \left| -\nabla PD^\epsilon(\hat{\mathbf{u}}^{k+1})(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^{k+1})(\mathbf{x}_i) \right|^2 \right)^{1/2}. \end{aligned}$$

From our consistency analysis, we have

$$\begin{aligned} \tau &= \sup_k \left( \|\tau^k(u)\|_{L^2(D; \mathbb{R}^d)} + \|\tau^k(v)\|_{L^2(D; \mathbb{R}^d)} \right. \\ &\quad \left. + \|\sigma^k(u)\|_{L^2(D; \mathbb{R}^d)} + \|\sigma^k(v)\|_{L^2(D; \mathbb{R}^d)} \right) \\ (48) \quad &\leq C_t \Delta t + C_s \frac{h^\gamma}{\epsilon^2}, \end{aligned}$$

where  $C_t$  and  $C_s$  are given by (37) and (38). Since  $0 \leq 1 - \theta \leq 1$  and  $0 \leq \theta \leq 1$  for all  $\theta \in [0, 1]$ , we have

$$\Theta (\|\tau^k(u)\| + \|\tau^k(v)\| + \|\sigma^k(u)\| + \|\sigma^k(v)\|) \leq 2\tau.$$

**Crank–Nicolson scheme.** If  $\theta = 1/2$  and if  $\mathbf{u}, \mathbf{v} \in C^3([0, T]; C^{0,\gamma}(D; \mathbb{R}^d))$ , then we can show that

$$\frac{1}{2}\tau_i^k(u) + \frac{1}{2}\tau_i^{k+1}(u) = \frac{(\Delta t)^2}{12} \frac{\partial^3 \tilde{\mathbf{u}}_i^{k+1/2}}{\partial t^3} + O((\Delta t)^3).$$

A similar result holds for  $1/2\tau_i^k(v) + 1/2\tau_i^{k+1}(v)$ . Therefore, the consistency error will be bounded by  $\bar{C}_t \Delta t^2 + C_s h^\gamma / \epsilon^2$  with

$$(49) \quad \bar{C}_t := \frac{1}{12} \sup_t \left\| \frac{\partial^3 \mathbf{u}(t)}{\partial t^3} \right\|_{L^2(D; \mathbb{R}^d)} + \frac{1}{12} \sup_t \left\| \frac{\partial^4 \mathbf{u}(t)}{\partial t^4} \right\|_{L^2(D; \mathbb{R}^d)},$$

and  $C_s$  is given by (38).

We now estimate (47). Similar to (40), we have

$$(50) \quad \left( \sum_{i, \mathbf{x}_i \in D} h^d \left| -\nabla PD^\epsilon(\hat{\mathbf{u}}^k)(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^k)(\mathbf{x}_i) \right|^2 \right)^{1/2} \leq \frac{\bar{C}}{\epsilon^2} \|\mathbf{e}^k(u)\|,$$

$$(51) \quad \left( \sum_{i, \mathbf{x}_i \in D} h^d \left| -\nabla PD^\epsilon(\hat{\mathbf{u}}^{k+1})(\mathbf{x}_i) + \nabla PD^\epsilon(\tilde{\mathbf{u}}^{k+1})(\mathbf{x}_i) \right|^2 \right)^{1/2} \leq \frac{\bar{C}}{\epsilon^2} \|\mathbf{e}^{k+1}(u)\|,$$

where  $\bar{C}$  is the constant given by (35). Let  $e^k := \|e^k(u)\| + \|e^k(v)\|$ . Adding (46) and (47) and noting (50), (51), and (48), we get

$$e^{k+1} \leq \left(1 + \Delta t(1 - \theta)\frac{\bar{C}}{\epsilon^2}\right)e^k + \Delta t\theta\frac{\bar{C}}{\epsilon^2}e^{k+1} + 2\tau\Delta t,$$

where we assumed  $\bar{C}/\epsilon^2 \geq 1$ . We further simplify the equation and write

$$(52) \quad e^{k+1} \leq \frac{1 + \Delta t(1 - \theta)\bar{C}/\epsilon^2}{1 - \Delta t\theta\bar{C}/\epsilon^2}e^k + \frac{2}{1 - \Delta t\theta\bar{C}/\epsilon^2}\tau\Delta t,$$

where we have assumed that  $1 - \Delta t\theta\bar{C}/\epsilon^2 > 0$ , i.e.,

$$(53) \quad \Delta t < \frac{\epsilon^2}{\bar{C}} = K\epsilon^2.$$

Thus, for fixed  $\epsilon > 0$ , the error calculation in this section applies when the time step  $\Delta t$  satisfies (53). We now define  $a$  and  $b$  by

$$a := \frac{1 + \Delta t(1 - \theta)\bar{C}/\epsilon^2}{1 - \Delta t\theta\bar{C}/\epsilon^2},$$

$$b := \frac{1}{1 - \Delta t\theta\bar{C}/\epsilon^2}.$$

We use the fact that, for  $\Delta t$  small,  $(1 - \alpha\Delta t)^{-1} = 1 + \alpha\Delta t + \alpha^2(\Delta t)^2 + O((\Delta t)^3)$ , to get

$$b = 1 + \Delta t\theta\bar{C}/\epsilon^2 + O\left((\Delta t/\epsilon^2)^2\right) = 1 + O(\Delta t/\epsilon^2).$$

Now since  $\Delta t < \epsilon^2/\bar{C}$ , we have

$$(54) \quad b = O(1).$$

We have the estimates for  $a$  given by

$$a \leq (1 + \Delta t(1 - \theta)\bar{C}/\epsilon^2)(1 + \Delta t\theta\bar{C}/\epsilon^2 + O((\Delta t/\epsilon^2)^2))$$

$$= 1 + \Delta t(\theta + (1 - \theta))\bar{C}/\epsilon^2 + O((\Delta t/\epsilon^2)^2)$$

$$= 1 + \Delta t\bar{C}/\epsilon^2 + O((\Delta t/\epsilon^2)^2).$$

Therefore, for any  $k \leq T/\Delta t$ , we have

$$a^k \leq \exp\left[k\Delta t\frac{\bar{C}}{\epsilon^2} + kO\left(\left(\frac{\Delta t}{\epsilon^2}\right)^2\right)\right]$$

$$\leq \exp\left[T\bar{C}/\epsilon^2 + O\left(\Delta t\left(\frac{1}{\epsilon^2}\right)^2\right)\right]$$

$$\leq \exp\left[TC/\epsilon^2 + O\left(\frac{1}{\epsilon^2}\right)\right],$$

where we simplified the bound by incorporating (53). Then, from (52), we get

$$e^{k+1} \leq a^{k+1}e^0 + 2\tau\left(\Delta t\sum_{j=0}^k a^j\right)b.$$

From the estimates on  $a^k$ , we have

$$(55) \quad \Delta t \sum_{j=0}^k a^j \leq T \exp[TC/\epsilon^2 + O(1/\epsilon^2)].$$

Combining (54) and (55), we get

$$e^k \leq \exp[TC/\epsilon^2 + O(1/\epsilon^2)] (e^0 + 2T\tau O(1)).$$

Since  $\tau = O(C_t \Delta t + C_s h^\gamma / \epsilon^2)$ , we conclude that, for any  $\epsilon > 0$  fixed,

$$\sup_k e^k \leq O(C_t \Delta t + C_s h^\gamma / \epsilon^2),$$

where we assumed  $e^0 = 0$ . Similarly, for  $\theta = 1/2$ , we have  $\sup_k e^{k+1} \leq O(\bar{C}_t (\Delta t)^2 + C_s h^\gamma / \epsilon^2)$ . Therefore, the scheme is stable and consistent for any  $\theta \in [0, 1]$ .

**3.3. Stability of the energy for the semidiscrete approximation.** We first spatially discretize the peridynamics equation (1). Let  $\{\hat{\mathbf{u}}_i(t)\}_{i, \mathbf{x}_i \in D}$  denote the semidiscrete approximate solution which satisfies the following: for all  $t \in [0, T]$  and  $i$  such that  $\mathbf{x}_i \in D$ ,

$$(56) \quad \ddot{\hat{\mathbf{u}}}_i(t) = -\nabla PD^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x}_i) + \mathbf{b}_i(t),$$

where  $\hat{\mathbf{u}}(t)$  is the piecewise constant extension of discrete set  $\{\hat{\mathbf{u}}_i(t)\}_i$  and is defined as

$$(57) \quad \hat{\mathbf{u}}(t, \mathbf{x}) := \sum_{i, \mathbf{x}_i \in D} \hat{\mathbf{u}}_i(t) \chi_{U_i}(\mathbf{x}).$$

The scheme is complemented with the discretized initial conditions  $\hat{\mathbf{u}}_i(0) = \mathbf{u}_0(\mathbf{x}_i)$  and  $\hat{\mathbf{v}}_i(0) = \mathbf{v}_0(\mathbf{x}_i)$ . We apply the boundary condition by setting  $\hat{\mathbf{u}}_i(t) = \mathbf{0}$  for all  $t$  and for all  $\mathbf{x}_i \notin D$ .

We have the stability of semidiscrete evolution.

**THEOREM 5** (energy stability of the semidiscrete approximation). *Let  $\{\hat{\mathbf{u}}_i(t)\}_i$  satisfy (56) and let  $\hat{\mathbf{u}}(t)$  be its piecewise constant extension. Similarly let  $\hat{\mathbf{b}}(t, \mathbf{x})$  denote the piecewise constant extension of  $\{\mathbf{b}(t, \mathbf{x}_i)\}_{i, \mathbf{x}_i \in D}$ . Then the peridynamic energy  $\mathcal{E}^\epsilon$  as defined in (4) satisfies, for all  $t \in [0, T]$ ,*

$$(58) \quad \mathcal{E}^\epsilon(\hat{\mathbf{u}})(t) \leq \left( \sqrt{\mathcal{E}^\epsilon(\hat{\mathbf{u}})(0)} + \frac{TC}{\epsilon^{3/2}} + \int_0^t \|\hat{\mathbf{b}}(s)\|_{L^2(D; \mathbb{R}^d)} ds \right)^2.$$

The constant  $C$ , defined in (63), is independent of  $\epsilon$  and  $h$ .

*Proof.* We multiply (56) by  $\chi_{U_i}(\mathbf{x})$  and sum over  $i$  and use the definition of piecewise constant extension in (57) to get

$$\begin{aligned} \ddot{\hat{\mathbf{u}}}(t, \mathbf{x}) &= -\nabla P \hat{D}^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x}) + \hat{\mathbf{b}}(t, \mathbf{x}) \\ &= -\nabla PD^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x}) + \hat{\mathbf{b}}(t, \mathbf{x}) \\ &\quad + (-\nabla P \hat{D}^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x}) + \nabla PD^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x})), \end{aligned}$$

where  $-\nabla PD^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x})$  and  $\hat{\mathbf{b}}(t, \mathbf{x})$  are given by

$$-\nabla PD^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x}) = \sum_{i, \mathbf{x}_i \in D} (-\nabla PD^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x}_i)) \chi_{U_i}(\mathbf{x}),$$

$$\hat{\mathbf{b}}(t, \mathbf{x}) = \sum_{i, \mathbf{x}_i \in D} \mathbf{b}(t, \mathbf{x}_i) \chi_{U_i}(\mathbf{x}).$$

We define the set as follows:

$$(59) \quad \sigma(t, \mathbf{x}) := -\nabla PD^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x}) + \nabla PD^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x}).$$

We use the following result which we will show after a few steps:

$$(60) \quad \|\sigma(t)\|_{L^2(D; \mathbb{R}^d)} \leq \frac{C}{\epsilon^{3/2}}.$$

We then have

$$(61) \quad \ddot{\mathbf{u}}(t, \mathbf{x}) = -\nabla PD^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x}) + \hat{\mathbf{b}}(t, \mathbf{x}) + \sigma(t, \mathbf{x}).$$

Multiplying above with  $\dot{\mathbf{u}}(t)$  and integrating over  $D$ , we get

$$(\ddot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) = (-\nabla PD^\epsilon(\hat{\mathbf{u}}(t)), \dot{\mathbf{u}}(t)) + (\hat{\mathbf{b}}(t), \dot{\mathbf{u}}(t)) + (\sigma(t), \dot{\mathbf{u}}(t)).$$

Consider energy  $\mathcal{E}^\epsilon(\hat{\mathbf{u}})(t)$  given by (4) and note the identity (5) in order to have

$$\begin{aligned} \frac{d}{dt} \mathcal{E}^\epsilon(\hat{\mathbf{u}})(t) &= (\hat{\mathbf{b}}(t), \dot{\mathbf{u}}(t)) + (\sigma(t), \dot{\mathbf{u}}(t)) \\ &\leq \left( \|\hat{\mathbf{b}}(t)\|_{L^2(D; \mathbb{R}^d)} + \|\sigma(t)\|_{L^2(D; \mathbb{R}^d)} \right) \|\dot{\mathbf{u}}(t)\|_{L^2(D; \mathbb{R}^d)}, \end{aligned}$$

where we used the Hölder inequality in the last step. Since  $PD^\epsilon(\mathbf{u})$  is positive for any  $\mathbf{u}$ , we have

$$\|\dot{\mathbf{u}}(t)\| \leq 2\sqrt{\frac{1}{2} \|\dot{\mathbf{u}}(t)\|_{L^2(D; \mathbb{R}^d)}^2 + PD^\epsilon(\hat{\mathbf{u}}(t))} = 2\sqrt{\mathcal{E}^\epsilon(\hat{\mathbf{u}})(t)}.$$

Using the above equation, we get

$$\frac{1}{2} \frac{d}{dt} \mathcal{E}^\epsilon(\hat{\mathbf{u}})(t) \leq \left( \|\hat{\mathbf{b}}(t)\|_{L^2(D; \mathbb{R}^d)} + \|\sigma(t)\|_{L^2(D; \mathbb{R}^d)} \right) \sqrt{\mathcal{E}^\epsilon(\hat{\mathbf{u}})(t)}.$$

Let  $\delta > 0$  be some arbitrary but fixed real number and let  $A(t) = \delta + \mathcal{E}^\epsilon(\hat{\mathbf{u}})(t)$ . Then

$$\frac{1}{2} \frac{d}{dt} A(t) \leq \left( \|\hat{\mathbf{b}}(t)\|_{L^2(D; \mathbb{R}^d)} + \|\sigma(t)\|_{L^2(D; \mathbb{R}^d)} \right) \sqrt{A(t)}.$$

Using the fact that  $\frac{1}{\sqrt{A(t)}} \frac{d}{dt} A(t) = 2 \frac{d}{dt} \sqrt{A(t)}$ , we have

$$\begin{aligned} \sqrt{A(t)} &\leq \sqrt{A(0)} + \int_0^t \left( \|\hat{\mathbf{b}}(s)\|_{L^2(D; \mathbb{R}^d)} + \|\sigma(s)\|_{L^2(D; \mathbb{R}^d)} \right) ds \\ &\leq \sqrt{A(0)} + \frac{TC}{\epsilon^{3/2}} + \int_0^T \|\hat{\mathbf{b}}(s)\|_{L^2(D; \mathbb{R}^d)} ds, \end{aligned}$$

where we used the bound on  $\|\sigma(s)\|_{L^2(D;\mathbb{R}^d)}$  from (60). Noting that  $\delta > 0$  is arbitrary, we send it to zero to get

$$\sqrt{\mathcal{E}^\epsilon(\hat{\mathbf{u}})(t)} \leq \sqrt{\mathcal{E}^\epsilon(\hat{\mathbf{u}})(0)} + \frac{TC}{\epsilon^{3/2}} + \int_0^T \|\hat{\mathbf{b}}(s)\| ds,$$

and (58) follows by taking the square of above equation.

It remains to show (60). To simplify the calculations, we use following notation: let  $\boldsymbol{\xi} \in H_1(\mathbf{0})$  and let

$$s_\xi = \epsilon|\boldsymbol{\xi}|, \quad e_\xi = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}, \quad \bar{\omega}(\mathbf{x}) = \omega(\mathbf{x})\omega(\mathbf{x} + \epsilon\boldsymbol{\xi}),$$

$$S_\xi(\mathbf{x}) = \frac{\hat{\mathbf{u}}(t, \mathbf{x} + \epsilon\boldsymbol{\xi}) - \hat{\mathbf{u}}(t, \mathbf{x})}{s_\xi} \cdot e_\xi.$$

With the above notation and using the expression of  $-\nabla PD^\epsilon$  from (73), we have for  $\mathbf{x} \in U_i$

$$\begin{aligned} |\sigma(t, \mathbf{x})| &= |-\nabla PD^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x}_i) + \nabla PD^\epsilon(\hat{\mathbf{u}}(t))(\mathbf{x})| \\ &= \left| \frac{2}{\epsilon\omega_d} \int_{H_1(\mathbf{0})} \frac{J(|\boldsymbol{\xi}|)}{\sqrt{s_\xi}} (\bar{\omega}(\mathbf{x}_i)F'_1(\sqrt{s_\xi}S_\xi(\mathbf{x}_i)) - \bar{\omega}(\mathbf{x})F'_1(\sqrt{s_\xi}S_\xi(\mathbf{x}))) e_\xi d\boldsymbol{\xi} \right| \\ &\leq \frac{2}{\epsilon\omega_d} \int_{H_1(\mathbf{0})} \frac{J(|\boldsymbol{\xi}|)}{\sqrt{s_\xi}} |\bar{\omega}(\mathbf{x}_i)F'_1(\sqrt{s_\xi}S_\xi(\mathbf{x}_i)) - \bar{\omega}(\mathbf{x})F'_1(\sqrt{s_\xi}S_\xi(\mathbf{x}))| d\boldsymbol{\xi} \\ (62) \quad &\leq \frac{2}{\epsilon\omega_d} \int_{H_1(\mathbf{0})} \frac{J(|\boldsymbol{\xi}|)}{\sqrt{s_\xi}} (|\bar{\omega}(\mathbf{x}_i)F'_1(\sqrt{s_\xi}S_\xi(\mathbf{x}_i))| + |\bar{\omega}(\mathbf{x})F'_1(\sqrt{s_\xi}S_\xi(\mathbf{x}))|) d\boldsymbol{\xi}. \end{aligned}$$

Using the fact that  $0 \leq \omega(\mathbf{x}) \leq 1$  and  $|F'_1(r)| \leq C_1$ , where  $C_1$  is  $\sup_r |F'_1(r)|$ , we get

$$|\sigma(t, \mathbf{x})| \leq \frac{4C_1\bar{J}_{1/2}}{\epsilon^{3/2}},$$

where  $\bar{J}_{1/2} = (1/\omega_d) \int_{H_1(\mathbf{0})} J(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^{-1/2} d\boldsymbol{\xi}$ .

Taking the  $L^2$  norm of  $\sigma(t, \mathbf{x})$ , we get

$$\|\sigma(t)\|_{L^2(D;\mathbb{R}^d)}^2 = \sum_{i, \mathbf{x}_i \in D} \int_{U_i} |\sigma(t, \mathbf{x})|^2 d\mathbf{x} \leq \left(\frac{4C_1\bar{J}_{1/2}}{\epsilon^{3/2}}\right)^2 \sum_{i, \mathbf{x}_i \in D} \int_{U_i} d\mathbf{x},$$

and thus

$$\|\sigma(t)\|_{L^2(D;\mathbb{R}^d)} \leq \frac{4C_1\bar{J}_{1/2}\sqrt{|D|}}{\epsilon^{3/2}} = \frac{C}{\epsilon^{3/2}},$$

where

$$(63) \quad C := 4C_1\bar{J}_{1/2}\sqrt{|D|}.$$

This completes the proof. □

**3.4. Local instability under radial perturbations.** We observe that both explicit and implicit schemes treated in previous sections show that any increase in local truncation error is controlled at each time step. From the proofs above (and the general approximation theory for ODEs), this control is adequate to establish convergence rates as  $\Delta t \rightarrow 0$ . We now comment on a source of error that can grow with time steps in regions where the strain is large and peridynamic bonds between material points begin to soften.

We examine the Jacobian matrix of the peridynamic system associated by perturbing about a displacement field and seek to understand the stability of the perturbation. Suppose the solution is near the displacement field  $\bar{\mathbf{u}}(\mathbf{x})$  and let  $\mathbf{s}(t, \mathbf{x}) = \mathbf{u}(t, \mathbf{x}) - \bar{\mathbf{u}}(\mathbf{x})$  be the perturbation. We write the associated strain as  $S(\mathbf{y}, \mathbf{x}; \bar{\mathbf{u}})$  and  $S(\mathbf{y}, \mathbf{x}; \mathbf{s})$ . Expanding the peridynamic force in Taylor series about  $\bar{\mathbf{u}}$  assuming  $\mathbf{s}$  is small gives

$$\begin{aligned} \partial_{tt}\mathbf{s}(t, \mathbf{x}) = & \frac{2}{V_d} \left\{ \int_{\mathcal{H}_\epsilon(\mathbf{x})} \partial_S^2 \mathcal{W}^\epsilon(S(\mathbf{y}, \mathbf{x}; \bar{\mathbf{u}})) S(\mathbf{y}, \mathbf{x}; \mathbf{s}) \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y} \right\} \\ & - \nabla PD^\epsilon(\bar{\mathbf{u}})(\mathbf{x}) + \mathbf{b}(t, \mathbf{x}) + O(|\mathbf{s}|^2), \end{aligned}$$

where  $\mathcal{W}^\epsilon(S, \mathbf{y} - \mathbf{x}) = W^\epsilon(S, \mathbf{y} - \mathbf{x})/|\mathbf{y} - \mathbf{x}|$  and  $W^\epsilon$  is given by (6).

To recover a local stability formula in terms of a spectral radius we consider local radial perturbations  $\mathbf{s}$  with spatially constant strain  $S(\mathbf{y}, \mathbf{x}; \mathbf{s})$  of the form  $S(\mathbf{y}, \mathbf{x}; \mathbf{s}) = -\delta(t)\boldsymbol{\mu} \cdot \mathbf{e}$ , where  $\boldsymbol{\mu}$  is in  $\mathbb{R}^d$  and  $\mathbf{s}$  has radial variation about  $\mathbf{x}$  with  $\mathbf{s}(\mathbf{y}) = \delta(t)\boldsymbol{\mu}(1 - |\mathbf{y} - \mathbf{x}|)$ . This delivers the local ODE

$$\delta''(t)\boldsymbol{\mu} = A\delta(t)\boldsymbol{\mu} + b,$$

where the stability matrix  $A$  is self-adjoint and given by

$$(64) \quad A = -\frac{2}{V_d} \left\{ \int_{\mathcal{H}_\epsilon(\mathbf{x})} \partial_S^2 \mathcal{W}^\epsilon(S(\mathbf{y}, \mathbf{x}; \bar{\mathbf{u}})) \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|} \otimes \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y} \right\}$$

and

$$b = -\nabla PD^\epsilon(\bar{\mathbf{u}})(\mathbf{x}) + \mathbf{b}(t, \mathbf{x}) + O(|\mathbf{s}|^2).$$

A stability criterion for the perturbation is obtained by analyzing the linear system  $\delta''(t)\boldsymbol{\mu} = A\delta(t)\boldsymbol{\mu}$ . Writing it as a first order system gives

$$\begin{aligned} \delta'_1(t)\boldsymbol{\mu} &= \delta_2(t)\boldsymbol{\mu}, \\ \delta'_2(t)\boldsymbol{\mu} &= A\delta_1(t)\boldsymbol{\mu}, \end{aligned}$$

where  $\boldsymbol{\mu}$  is a vector in  $\mathbb{R}^d$ . The eigenvalues of  $A$  are real and denoted by  $\lambda_i$ ,  $i = 1, \dots, d$ , and the associated eigenvectors are denoted by  $\mathbf{v}^i$ . Choosing  $\boldsymbol{\mu} = \mathbf{v}^i$  gives

$$\begin{aligned} \delta'_1(t)\mathbf{v}^i &= \delta_2(t)\mathbf{v}^i, \\ \delta'_2(t)\mathbf{v}^i &= \lambda_i\delta_1(t)\mathbf{v}^i. \end{aligned}$$

Applying the forward Euler method to this system gives the discrete iterative system

$$\begin{aligned} \delta_1^{k+1} &= \delta_1^k + \Delta t\delta_2^k, \\ \delta_2^{k+1} &= \lambda_i\Delta t\delta_1^k + \delta_2^k. \end{aligned}$$

The spectral radius of the matrix associated with this iteration is

$$\rho = \max_{i=1,\dots,d} |1 \pm \Delta t \sqrt{\lambda_i}|.$$

It is easy to see that the spectral radius is larger than 1 for any choice of  $\lambda_i$  and we conclude local instability for the forward Euler scheme under radial perturbation.

For the implicit scheme given by backward Euler we get the discrete iterative system

$$\begin{aligned} \delta_1^k &= \delta_1^{k+1} - \Delta t \delta_2^{k+1}, \\ \delta_2^k &= -\lambda_i \Delta t \delta_1^{k+1} + \delta_2^{k+1}, \end{aligned}$$

and

$$\begin{bmatrix} \delta_1^{k+1} \\ \delta_2^{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -\Delta t \\ -\Delta t \lambda_i & 1 \end{bmatrix}^{-1} \begin{bmatrix} \delta_1^k \\ \delta_2^k \end{bmatrix}.$$

The spectral radius for the iteration matrix is

$$\rho = \max_{i=1,\dots,d} \left| \frac{1}{\theta} \pm \frac{\Delta t \sqrt{\lambda_i}}{|\theta|} \right|,$$

where  $\theta = 1 - \lambda_i(\Delta t)^2$ . If we suppose that the stability matrix  $A$  is not negative definite and there is a  $\lambda_j > 0$ , then the spectral radius is larger than one, i.e.,

$$(65) \quad 1 < \frac{|1 + \sqrt{\lambda_j} \Delta t|}{|1 - \lambda_j(\Delta t)^2|} \leq \rho.$$

Thus it follows from (65) that we can have local instability of the backward Euler scheme for radial perturbations. Inspection of (64) shows the sign of the eigenvalues of the matrix  $A$  depend explicitly on the sign of  $\partial_S^2 \mathcal{W}^\epsilon(S(\mathbf{y}, \mathbf{x}; \bar{\mathbf{u}}))$ . It is shown in [23] that

$$(66) \quad \partial_S^2 \mathcal{W}^\epsilon(S(\mathbf{y}, \mathbf{x}; \bar{\mathbf{u}})) > 0 \text{ for } |S(\mathbf{y}, \mathbf{x}; \bar{\mathbf{u}})| < S_c,$$

$$(67) \quad \partial_S^2 \mathcal{W}^\epsilon(S(\mathbf{y}, \mathbf{x}; \bar{\mathbf{u}})) < 0 \text{ for } |S(\mathbf{y}, \mathbf{x}; \bar{\mathbf{u}})| > S_c.$$

From the model we see that bonds are losing stiffness when  $|S(\mathbf{y}, \mathbf{x}; \bar{\mathbf{u}})| > S_c$  and the points for which  $A$  is nonnegative definite correspond to points where (67) holds for a preponderance of bonds inside the horizon. We conclude noting that both explicit and implicit schemes treated in previous sections have demonstrated convergence rates  $O((C_t \Delta t + C_s h^\gamma / \epsilon^2))$  as  $\Delta t \rightarrow 0$ . However, the results of this section show that the error can grow with time for this type of radial perturbation.

**4. Lipschitz continuity in Hölder norm and existence of a solution.**

In this section, we prove Theorems 1, 6, and 2.

**4.1. Proof of Theorem 1.** Let  $I = [0, T]$  be the time domain and let  $X = C_0^{0,\gamma}(D; \mathbb{R}^d) \times C_0^{0,\gamma}(D; \mathbb{R}^d)$ . Recall that  $F^\epsilon(y, t) = (F_1^\epsilon(y, t), F_2^\epsilon(y, t))$ , where  $F_1^\epsilon(y, t) = y^2$  and  $F_2^\epsilon(y, t) = -\nabla P D^\epsilon(y^1) + \mathbf{b}(t)$ . Given  $t \in I$  and  $y = (y^1, y^2), z = (z^1, z^2) \in X$ , we have

$$(68) \quad \begin{aligned} & \|F^\epsilon(y, t) - F^\epsilon(z, t)\|_X \\ & \leq \|y^2 - z^2\|_{C^{0,\gamma}(D; \mathbb{R}^d)} + \|-\nabla P D^\epsilon(y^1) + \nabla P D^\epsilon(z^1)\|_{C^{0,\gamma}(D; \mathbb{R}^d)}. \end{aligned}$$

Therefore, to prove (11), we need only analyze the second term in the above inequality. Let  $\mathbf{u}, \mathbf{v} \in C^{0,\gamma}(D; \mathbb{R}^d)$ ; then we have

$$\begin{aligned}
 & \|-\nabla PD^\epsilon(\mathbf{u}) - (-\nabla PD^\epsilon(\mathbf{v}))\|_{C^{0,\gamma}(D; \mathbb{R}^d)} \\
 &= \sup_{\mathbf{x} \in D} |-\nabla PD^\epsilon(\mathbf{u})(\mathbf{x}) - (-\nabla PD^\epsilon(\mathbf{v})(\mathbf{x}))| \\
 (69) \quad &+ \sup_{\substack{\mathbf{x} \neq \mathbf{y}, \\ \mathbf{x}, \mathbf{y} \in D}} \frac{|(-\nabla PD^\epsilon(\mathbf{u}) + \nabla PD^\epsilon(\mathbf{v}))(\mathbf{x}) - (-\nabla PD^\epsilon(\mathbf{u}) + \nabla PD^\epsilon(\mathbf{v}))(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\gamma}.
 \end{aligned}$$

Note that the force  $-\nabla PD^\epsilon(\mathbf{u})(\mathbf{x})$  can be written as follows:

$$\begin{aligned}
 & -\nabla PD^\epsilon(\mathbf{u})(\mathbf{x}) \\
 &= \frac{4}{\epsilon^{d+1}\omega_d} \int_{H_\epsilon(\mathbf{x})} \omega(\mathbf{x})\omega(\mathbf{y})J\left(\frac{|\mathbf{y} - \mathbf{x}|}{\epsilon}\right) f'(|\mathbf{y} - \mathbf{x}| S(\mathbf{y}, \mathbf{x}; \mathbf{u})^2) S(\mathbf{y}, \mathbf{x}; \mathbf{u}) \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y} \\
 &= \frac{4}{\epsilon\omega_d} \int_{H_1(\mathbf{0})} \omega(\mathbf{x})\omega(\mathbf{x} + \epsilon\xi)J(|\xi|)f'(\epsilon|\xi| S(\mathbf{x} + \epsilon\xi, \mathbf{x}; \mathbf{u})^2) S(\mathbf{x} + \epsilon\xi, \mathbf{x}; \mathbf{u}) \frac{\xi}{|\xi|} d\xi,
 \end{aligned}$$

where we substituted  $\partial_S W^\epsilon$  using (6). In the second step, we introduced the change in the variable  $\mathbf{y} = \mathbf{x} + \epsilon\xi$ .

Let  $F_1 : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $F_1(S) = f(S^2)$ . Then  $F_1'(S) = f'(S^2)2S$ . Using the definition of  $F_1$ , we have

$$2Sf'(\epsilon|\xi| S^2) = \frac{F_1'(\sqrt{\epsilon|\xi|}S)}{\sqrt{\epsilon|\xi|}}.$$

Because  $f$  is assumed to be positive, smooth, and concave, and is bounded far away, we have following bound on derivatives of  $F_1$ :

$$\begin{aligned}
 (70) \quad & \sup_r |F_1'(r)| = F_1'(\bar{r}) =: C_1, \\
 (71) \quad & \sup_r |F_1''(r)| = \max\{F_1''(0), F_1''(\hat{u})\} =: C_2, \\
 (72) \quad & \sup_r |F_1'''(r)| = \max\{F_1'''(\bar{u}_2), F_1'''(\tilde{u}_2)\} =: C_3,
 \end{aligned}$$

where  $\bar{r}$  is the inflection point of  $f(r^2)$ , i.e.,  $F_1''(\bar{r}) = 0$ .  $\{0, \hat{u}\}$  are the maxima of  $F_1''(r)$ .  $\{\bar{u}, \tilde{u}\}$  are the maxima of  $F_1'''(r)$ . By the chain rule and by considering the assumption on  $f$ , we can show that  $\bar{r}, \hat{u}, \bar{u}_2, \tilde{u}_2$  exists and the  $C_1, C_2, C_3$  are bounded. Figures 4, 5, and 6 show the generic graphs of  $F_1'(r)$ ,  $F_1''(r)$ , and  $F_1'''(r)$ , respectively.

The nonlocal force  $-\nabla PD^\epsilon$  can be written as

$$\begin{aligned}
 & -\nabla PD^\epsilon(\mathbf{u})(\mathbf{x}) \\
 (73) \quad &= \frac{2}{\epsilon\omega_d} \int_{H_1(\mathbf{0})} \omega(\mathbf{x})\omega(\mathbf{x} + \epsilon\xi)J(|\xi|)F_1'(\sqrt{\epsilon|\xi|}S(\mathbf{x} + \epsilon\xi, \mathbf{x}; \mathbf{u})) \frac{1}{\sqrt{\epsilon|\xi|}} \frac{\xi}{|\xi|} d\xi.
 \end{aligned}$$

To simplify the calculations, we use following notation:

$$\begin{aligned}
 \bar{\mathbf{u}}(\mathbf{x}) &:= \mathbf{u}(\mathbf{x} + \epsilon\xi) - \mathbf{u}(\mathbf{x}), \\
 \bar{\mathbf{u}}(\mathbf{y}) &:= \mathbf{u}(\mathbf{y} + \epsilon\xi) - \mathbf{u}(\mathbf{y}), \\
 (\mathbf{u} - \mathbf{v})(\mathbf{x}) &:= \mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x}),
 \end{aligned}$$



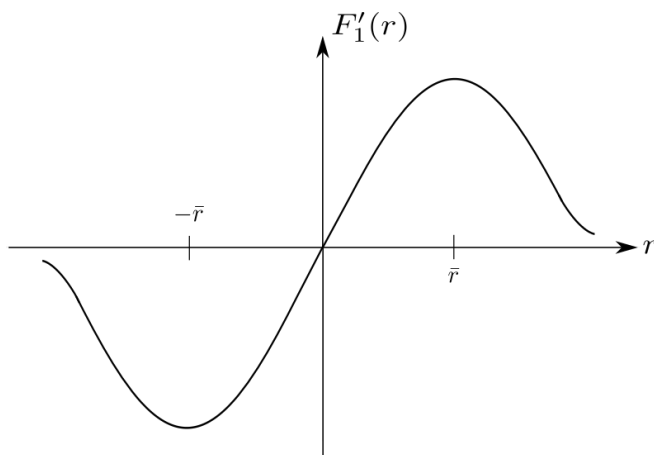


FIG. 4. Generic plot of  $F'_1(r)$ .  $|F'_1(r)|$  is bounded by  $|F'_1(\bar{r})|$ .

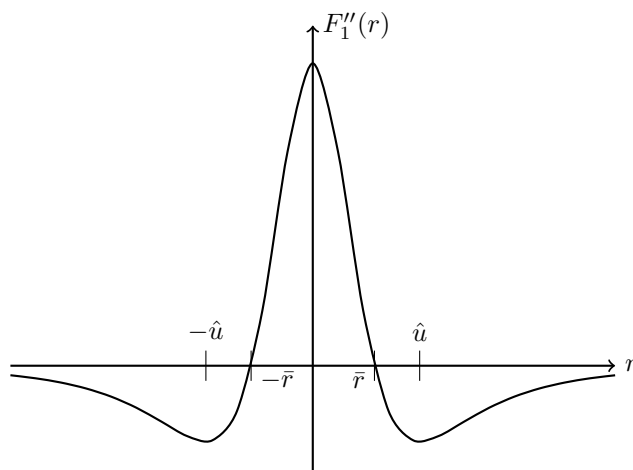


FIG. 5. Generic plot of  $F''_1(r)$ . At  $\pm\bar{r}$ ,  $F''_1(r) = 0$ . At  $\pm\hat{u}$ ,  $F''_1(r) = 0$ .

and  $\overline{(\mathbf{u} - \mathbf{v})}(\mathbf{x})$  is defined similarly to  $\bar{\mathbf{u}}(\mathbf{x})$ . Also, let

$$s = \epsilon |\boldsymbol{\xi}|, \quad \mathbf{e} = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}.$$

In what follows, we will come across the integral of type  $\int_{H_1(\mathbf{0})} J(|\boldsymbol{\xi}|) |\boldsymbol{\xi}|^{-\alpha} d\boldsymbol{\xi}$ . Recall that  $0 \leq J(|\boldsymbol{\xi}|) \leq M$  for all  $\boldsymbol{\xi} \in H_1(\mathbf{0})$  and  $J(|\boldsymbol{\xi}|) = 0$  for  $\boldsymbol{\xi} \notin H_1(\mathbf{0})$ . Therefore, let

$$(74) \quad \bar{J}_\alpha := \frac{1}{\omega_d} \int_{H_1(\mathbf{0})} J(|\boldsymbol{\xi}|) |\boldsymbol{\xi}|^{-\alpha} d\boldsymbol{\xi}.$$

With the notation above, we note that  $S(\mathbf{x} + \epsilon\boldsymbol{\xi}, \mathbf{x}; \mathbf{u}) = \bar{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{e}/s$ .  $-\nabla PD^\epsilon$  can be written as

$$(75) \quad -\nabla PD^\epsilon(\mathbf{u})(\mathbf{x}) = \frac{2}{\epsilon\omega_d} \int_{H_1(\mathbf{0})} \omega(\mathbf{x})\omega(\mathbf{x} + \epsilon\boldsymbol{\xi})J(|\boldsymbol{\xi}|)F'_1(\bar{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{e}/\sqrt{s})\frac{1}{\sqrt{s}}\mathbf{e}d\boldsymbol{\xi}.$$

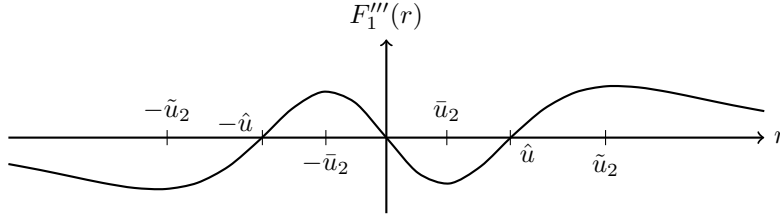


FIG. 6. Generic plot of  $F_1'''(r)$ . At  $\pm\bar{u}_2$  and  $\pm\tilde{u}_2$ ,  $F_1'''' = 0$ .

We first estimate the term  $|-\nabla PD^\epsilon(\mathbf{u})(\mathbf{x}) - (-\nabla PD^\epsilon(\mathbf{v})(\mathbf{x}))|$  in (69). We then have

$$\begin{aligned}
 & |-\nabla PD^\epsilon(\mathbf{u})(\mathbf{x}) - (-\nabla PD^\epsilon(\mathbf{v})(\mathbf{x}))| \\
 & \leq \left| \frac{2}{\epsilon\omega_d} \int_{H_1(\mathbf{0})} \omega(\mathbf{x})\omega(\mathbf{x} + \epsilon\xi)J(|\xi|) \frac{(F_1'(\bar{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{e}/\sqrt{s}) - F_1'(\bar{\mathbf{v}}(\mathbf{x}) \cdot \mathbf{e}/\sqrt{s}))}{\sqrt{s}} \epsilon d\xi \right| \\
 & \leq \left| \frac{2}{\epsilon\omega_d} \int_{H_1(\mathbf{0})} J(|\xi|) \frac{1}{\sqrt{s}} |F_1'(\bar{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{e}/\sqrt{s}) - F_1'(\bar{\mathbf{v}}(\mathbf{x}) \cdot \mathbf{e}/\sqrt{s})| d\xi \right| \\
 & \leq \sup_r |F_1''(r)| \left| \frac{2}{\epsilon\omega_d} \int_{H_1(\mathbf{0})} J(|\xi|) \frac{1}{\sqrt{s}} |\bar{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{e}/\sqrt{s} - \bar{\mathbf{v}}(\mathbf{x}) \cdot \mathbf{e}/\sqrt{s}| d\xi \right| \\
 (76) \quad & \leq \frac{2C_2}{\epsilon\omega_d} \left| \int_{H_1(\mathbf{0})} J(|\xi|) \frac{|\bar{\mathbf{u}}(\mathbf{x}) - \bar{\mathbf{v}}(\mathbf{x})|}{\epsilon|\xi|} d\xi \right|.
 \end{aligned}$$

Here we have used the fact that  $|\omega(\mathbf{x})| \leq 1$ , and for a vector  $\mathbf{e}$  such that  $|\mathbf{e}| = 1$ ,  $|\mathbf{a} \cdot \mathbf{e}| \leq |\mathbf{a}|$  holds and  $|\alpha\mathbf{e}| \leq |\alpha|$  holds for all  $\mathbf{a} \in \mathbb{R}^d, \alpha \in \mathbb{R}$ . Using the fact that  $\mathbf{u}, \mathbf{v} \in C_0^{0,\gamma}(D; \mathbb{R}^d)$ , we have

$$\begin{aligned}
 \frac{|\bar{\mathbf{u}}(\mathbf{x}) - \bar{\mathbf{v}}(\mathbf{x})|}{s} &= \frac{|(\mathbf{u} - \mathbf{v})(\mathbf{x} + \epsilon\xi) - (\mathbf{u} - \mathbf{v})(\mathbf{x})|}{(\epsilon|\xi|)^\gamma} \frac{1}{(\epsilon|\xi|)^{1-\gamma}} \\
 &\leq \|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}(D; \mathbb{R}^d)} \frac{1}{(\epsilon|\xi|)^{1-\gamma}}.
 \end{aligned}$$

Substituting the estimate given above, we get

$$(77) \quad |-\nabla PD^\epsilon(\mathbf{u})(\mathbf{x}) - (-\nabla PD^\epsilon(\mathbf{v})(\mathbf{x}))| \leq \frac{2C_2\bar{J}_{1-\gamma}}{\epsilon^{2-\gamma}} \|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}(D; \mathbb{R}^d)},$$

where  $C_2$  is given by (71) and  $\bar{J}_{1-\gamma}$  is given by (74).

We now estimate the second term in (69). To simplify notation, we write  $\tilde{\omega}(\mathbf{x}, \xi) =$

$\omega(\mathbf{x})\omega(\mathbf{x} + \epsilon\xi)$  and with the help of (75), we get

$$\begin{aligned}
 & \frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} |(-\nabla PD^\epsilon(\mathbf{u}) + \nabla PD^\epsilon(\mathbf{v}))(\mathbf{x}) - (-\nabla PD^\epsilon(\mathbf{u}) + \nabla PD^\epsilon(\mathbf{v}))(\mathbf{y})| \\
 &= \frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} \left| \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \frac{1}{\sqrt{s}} \times \left( \tilde{\omega}(\mathbf{x}, \xi) \left( F_1' \left( \frac{\bar{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{e}}{\sqrt{s}} \right) - F_1' \left( \frac{\bar{\mathbf{v}}(\mathbf{x}) \cdot \mathbf{e}}{\sqrt{s}} \right) \right) \right. \right. \\
 & \quad \left. \left. - \tilde{\omega}(\mathbf{y}, \xi) \left( F_1' \left( \frac{\bar{\mathbf{u}}(\mathbf{y}) \cdot \mathbf{e}}{\sqrt{s}} \right) - F_1' \left( \frac{\bar{\mathbf{v}}(\mathbf{y}) \cdot \mathbf{e}}{\sqrt{s}} \right) \right) \right) \right| d\xi \\
 &\leq \frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} \left| \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \frac{1}{\sqrt{s}} \right. \\
 & \quad \left. \times |\tilde{\omega}(\mathbf{x}, \xi) \left( F_1' \left( \frac{\bar{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{e}}{\sqrt{s}} \right) - F_1' \left( \frac{\bar{\mathbf{v}}(\mathbf{x}) \cdot \mathbf{e}}{\sqrt{s}} \right) \right) - \tilde{\omega}(\mathbf{y}, \xi) \left( F_1' \left( \frac{\bar{\mathbf{u}}(\mathbf{y}) \cdot \mathbf{e}}{\sqrt{s}} \right) - F_1' \left( \frac{\bar{\mathbf{v}}(\mathbf{y}) \cdot \mathbf{e}}{\sqrt{s}} \right) \right)| d\xi. \tag{78}
 \end{aligned}$$

We analyze the integrand in the above equation. We let  $H$  be defined by

$$H := \frac{\left| \tilde{\omega}(\mathbf{x}, \xi) \left( F_1' \left( \frac{\bar{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{e}}{\sqrt{s}} \right) - F_1' \left( \frac{\bar{\mathbf{v}}(\mathbf{x}) \cdot \mathbf{e}}{\sqrt{s}} \right) \right) - \tilde{\omega}(\mathbf{y}, \xi) \left( F_1' \left( \frac{\bar{\mathbf{u}}(\mathbf{y}) \cdot \mathbf{e}}{\sqrt{s}} \right) - F_1' \left( \frac{\bar{\mathbf{v}}(\mathbf{y}) \cdot \mathbf{e}}{\sqrt{s}} \right) \right) \right|}{|\mathbf{x} - \mathbf{y}|^\gamma}.$$

Let  $\mathbf{r} : [0, 1] \times D \rightarrow \mathbb{R}^d$  be defined as

$$\mathbf{r}(l, \mathbf{x}) = \bar{\mathbf{v}}(\mathbf{x}) + l(\bar{\mathbf{u}}(\mathbf{x}) - \bar{\mathbf{v}}(\mathbf{x})).$$

Note  $\partial \mathbf{r}(l, \mathbf{x}) / \partial l = \bar{\mathbf{u}}(\mathbf{x}) - \bar{\mathbf{v}}(\mathbf{x})$ . Using  $\mathbf{r}(l, \mathbf{x})$ , we have

$$(79) \quad F_1'(\bar{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{e} / \sqrt{s}) - F_1'(\bar{\mathbf{v}}(\mathbf{x}) \cdot \mathbf{e} / \sqrt{s}) = \int_0^1 \frac{\partial F_1'(\mathbf{r}(l, \mathbf{x}) \cdot \mathbf{e} / \sqrt{s})}{\partial l} dl$$

$$(80) \quad = \int_0^1 \frac{\partial F_1'(\mathbf{r} \cdot \mathbf{e} / \sqrt{s})}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}(l, \mathbf{x})} \cdot \frac{\partial \mathbf{r}(l, \mathbf{x})}{\partial l} dl.$$

Similarly, we have

$$(81) \quad F_1'(\bar{\mathbf{u}}(\mathbf{y}) \cdot \mathbf{e} / \sqrt{s}) - F_1'(\bar{\mathbf{v}}(\mathbf{y}) \cdot \mathbf{e} / \sqrt{s}) = \int_0^1 \frac{\partial F_1'(\mathbf{r} \cdot \mathbf{e} / \sqrt{s})}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}(l, \mathbf{y})} \cdot \frac{\partial \mathbf{r}(l, \mathbf{y})}{\partial l} dl.$$

Note that

$$(82) \quad \frac{\partial F_1'(\mathbf{r} \cdot \mathbf{e} / \sqrt{s})}{\partial \mathbf{r}} \Big|_{\mathbf{r}=\mathbf{r}(l, \mathbf{y})} = F_1''(\mathbf{r}(l, \mathbf{x}) \cdot \mathbf{e} / \sqrt{s}) \frac{\mathbf{e}}{\sqrt{s}}.$$

Combining (80), (81), and (82) gives

$$\begin{aligned}
 H &= \frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} \left| \int_0^1 (\tilde{\omega}(\mathbf{x}, \xi) F_1''(\mathbf{r}(l, \mathbf{x}) \cdot \mathbf{e} / \sqrt{s})(\bar{\mathbf{u}}(\mathbf{x}) - \bar{\mathbf{v}}(\mathbf{x})) \right. \\
 & \quad \left. - \tilde{\omega}(\mathbf{y}, \xi) F_1''(\mathbf{r}(l, \mathbf{y}) \cdot \mathbf{e} / \sqrt{s})(\bar{\mathbf{u}}(\mathbf{y}) - \bar{\mathbf{v}}(\mathbf{y})) \right) \cdot \frac{\mathbf{e}}{\sqrt{s}} dl \Big| \\
 &\leq \frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} \frac{1}{\sqrt{s}} \left| \int_0^1 |\tilde{\omega}(\mathbf{x}, \xi) F_1''(\mathbf{r}(l, \mathbf{x}) \cdot \mathbf{e} / \sqrt{s})(\bar{\mathbf{u}}(\mathbf{x}) - \bar{\mathbf{v}}(\mathbf{x})) \right. \\
 & \quad \left. - \tilde{\omega}(\mathbf{y}, \xi) F_1''(\mathbf{r}(l, \mathbf{y}) \cdot \mathbf{e} / \sqrt{s})(\bar{\mathbf{u}}(\mathbf{y}) - \bar{\mathbf{v}}(\mathbf{y}))| dl \right|.
 \end{aligned}$$

Adding and subtracting  $\tilde{\omega}(\mathbf{x}, \boldsymbol{\xi})F_1''(\mathbf{r}(l, \mathbf{x}) \cdot \mathbf{e}/\sqrt{s})(\bar{\mathbf{u}}(\mathbf{y}) - \bar{\mathbf{v}}(\mathbf{y}))$ , and noting  $0 \leq \tilde{\omega}(\mathbf{x}, \boldsymbol{\xi}) \leq 1$ , gives

$$\begin{aligned} H &\leq \frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} \frac{1}{\sqrt{s}} \left| \int_0^1 |F_1''(\mathbf{r}(l, \mathbf{x}) \cdot \mathbf{e}/\sqrt{s})| |\bar{\mathbf{u}}(\mathbf{x}) - \bar{\mathbf{v}}(\mathbf{x}) - \bar{\mathbf{u}}(\mathbf{y}) + \bar{\mathbf{v}}(\mathbf{y})| dl \right| \\ &\quad + \frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} \frac{1}{\sqrt{s}} \int_0^1 |(\tilde{\omega}(\mathbf{x}, \boldsymbol{\xi})F_1''(\mathbf{r}(l, \mathbf{x}) \cdot \mathbf{e}/\sqrt{s}) - \tilde{\omega}(\mathbf{y}, \boldsymbol{\xi})F_1''(\mathbf{r}(l, \mathbf{y}) \cdot \mathbf{e}/\sqrt{s}))| \\ &\quad \quad \quad \times |\bar{\mathbf{u}}(\mathbf{y}) - \bar{\mathbf{v}}(\mathbf{y})| dl \\ &=: H_1 + H_2. \end{aligned}$$

We estimate  $H_1$  first. Note that  $|F_1''(r)| \leq C_2$ . Since  $\mathbf{u}, \mathbf{v} \in C_0^{0,\gamma}(D; \mathbb{R}^d)$ , it is easily seen that

$$\frac{|\bar{\mathbf{u}}(\mathbf{x}) - \bar{\mathbf{v}}(\mathbf{x}) - \bar{\mathbf{u}}(\mathbf{y}) + \bar{\mathbf{v}}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\gamma} \leq 2\|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}(D; \mathbb{R}^d)}.$$

Therefore, we have

$$(83) \quad H_1 \leq \frac{2C_2}{\sqrt{s}} \|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}(D; \mathbb{R}^d)}.$$

We now estimate  $H_2$ . We add and subtract  $\tilde{\omega}(\mathbf{x}, \boldsymbol{\xi})F_1''(\mathbf{r}(l, \mathbf{y}) \cdot \mathbf{e}/\sqrt{s})$  in  $H_2$  to get

$$H_2 \leq H_3 + H_4,$$

where

$$H_3 = \frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} \frac{1}{\sqrt{s}} \int_0^1 |(F_1''(\mathbf{r}(l, \mathbf{x}) \cdot \mathbf{e}/\sqrt{s}) - F_1''(\mathbf{r}(l, \mathbf{y}) \cdot \mathbf{e}/\sqrt{s}))| |\bar{\mathbf{u}}(\mathbf{y}) - \bar{\mathbf{v}}(\mathbf{y})| dl$$

and

$$H_4 = \frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} \frac{1}{\sqrt{s}} \int_0^1 |(\tilde{\omega}(\mathbf{x}, \boldsymbol{\xi}) - \tilde{\omega}(\mathbf{y}, \boldsymbol{\xi}))F_1''(\mathbf{r}(l, \mathbf{y}) \cdot \mathbf{e}/\sqrt{s})| |\bar{\mathbf{u}}(\mathbf{y}) - \bar{\mathbf{v}}(\mathbf{y})| dl.$$

Now we estimate  $H_3$ . Since  $|F_1'''(r)| \leq C_3$  (see (72)), we have

$$\begin{aligned} &\frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} |F_1''(\mathbf{r}(l, \mathbf{x}) \cdot \mathbf{e}/\sqrt{s}) - F_1''(\mathbf{r}(l, \mathbf{y}) \cdot \mathbf{e}/\sqrt{s})| \\ &\leq \frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} \sup_r |F_1'''(r)| \frac{|\mathbf{r}(l, \mathbf{x}) \cdot \mathbf{e} - \mathbf{r}(l, \mathbf{y}) \cdot \mathbf{e}|}{\sqrt{s}} \\ &\leq \frac{C_3}{\sqrt{s}} \frac{|\mathbf{r}(l, \mathbf{x}) - \mathbf{r}(l, \mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\gamma} \\ &= \frac{C_3}{\sqrt{s}} \left( \frac{|1-l||\bar{\mathbf{v}}(\mathbf{x}) - \bar{\mathbf{v}}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\gamma} + \frac{|l||\bar{\mathbf{u}}(\mathbf{x}) - \bar{\mathbf{u}}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\gamma} \right) \\ (84) \quad &\leq \frac{C_3}{\sqrt{s}} \left( \frac{|\bar{\mathbf{v}}(\mathbf{x}) - \bar{\mathbf{v}}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\gamma} + \frac{|\bar{\mathbf{u}}(\mathbf{x}) - \bar{\mathbf{u}}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\gamma} \right), \end{aligned}$$

where we have used the fact that  $|1 - l| \leq 1, |l| \leq 1$ , as  $l \in [0, 1]$ . Also, note that

$$\begin{aligned} \frac{|\bar{\mathbf{u}}(\mathbf{x}) - \bar{\mathbf{u}}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\gamma} &\leq 2\|\mathbf{u}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}, \\ \frac{|\bar{\mathbf{v}}(\mathbf{x}) - \bar{\mathbf{v}}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\gamma} &\leq 2\|\mathbf{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}, \\ |\bar{\mathbf{u}}(\mathbf{y}) - \bar{\mathbf{v}}(\mathbf{y})| &\leq s^\gamma\|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \end{aligned}$$

We combine the above estimates with (84) to get

$$\begin{aligned} H_3 &\leq \frac{1}{\sqrt{s}} \frac{C_3}{\sqrt{s}} \left( \|\mathbf{u}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} + \|\mathbf{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \right) s^\gamma \|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \\ (85) \quad &= \frac{C_3}{s^{1-\gamma}} \left( \|\mathbf{u}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} + \|\mathbf{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \right) \|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \end{aligned}$$

Next we estimate  $H_4$ . Here we add and subtract  $\omega(\mathbf{y})\omega(\mathbf{x} + \epsilon\xi)$  to get

$$\begin{aligned} H_4 &= \frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} \frac{1}{\sqrt{s}} \int_0^1 |(\omega(\mathbf{x}, \mathbf{x} + \epsilon\xi)(\omega(\mathbf{x}) - \omega(\mathbf{y})) + \omega(\mathbf{y})(\omega(\mathbf{x} + \epsilon\xi) - \omega(\mathbf{y} + \epsilon\xi))) \\ &\quad \times |F_1''(\mathbf{r}(l, \mathbf{y}) \cdot \mathbf{e}/\sqrt{s})| |\bar{\mathbf{u}}(\mathbf{y}) - \bar{\mathbf{v}}(\mathbf{y})| dl. \end{aligned}$$

Recalling that  $\omega$  belongs to  $C^{0,\gamma}(D;\mathbb{R}^d)$  and in view of the previous estimates, a straightforward calculation gives

$$(86) \quad H_4 \leq \frac{4C_2}{s^{1/2-\gamma}} \|\omega\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}.$$

Combining (83), (85), and (86) gives

$$\begin{aligned} H &\leq \left( \frac{2C_2}{\sqrt{s}} + \frac{4C_2}{s^{1/2-\gamma}} \|\omega\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \right. \\ &\quad \left. + \frac{C_3}{s^{1-\gamma}} \left( \|\mathbf{u}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} + \|\mathbf{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \right) \right) \|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \end{aligned}$$

Substituting  $H$  in (78) gives

$$\begin{aligned} &\frac{1}{|\mathbf{x} - \mathbf{y}|^\gamma} |(-\nabla PD^\epsilon(\mathbf{u}) + \nabla PD^\epsilon(\mathbf{v}))(\mathbf{x}) - (-\nabla PD^\epsilon(\mathbf{u}) + \nabla PD^\epsilon(\mathbf{v}))(\mathbf{y})| \\ &\leq \left| \frac{2}{\epsilon\omega_d} \int_{H_1(\mathbf{0})} J(|\xi|) \frac{1}{\sqrt{s}} H d\xi \right| \\ &\leq \left( \frac{4C_2\bar{J}_1}{\epsilon^2} + \frac{4C_2\bar{J}_{1-\gamma}}{\epsilon^{2-\gamma}} \|\omega\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \right. \\ (87) \quad &\left. + \frac{2C_3\bar{J}_{3/2-\gamma}}{\epsilon^{2+1/2-\gamma}} \left( \|\mathbf{u}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} + \|\mathbf{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)} \right) \right) \|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}(D;\mathbb{R}^d)}. \end{aligned}$$

We combine (69), (77), and (87), and get

$$\begin{aligned} (88) \quad &\|-\nabla PD^\epsilon(\mathbf{u}) - (-\nabla PD^\epsilon(\mathbf{v}))\|_{C^{0,\gamma}} \\ &\leq \left( \frac{4C_2\bar{J}_1}{\epsilon^2} + \frac{2C_2\bar{J}_{1-\gamma}}{\epsilon^{2-\gamma}} (1 + \|\omega\|_{C^{0,\gamma}}) + \frac{2C_3\bar{J}_{3/2-\gamma}}{\epsilon^{2+1/2-\gamma}} (\|\mathbf{u}\|_{C^{0,\gamma}} + \|\mathbf{v}\|_{C^{0,\gamma}}) \right) \|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}} \\ &\leq \frac{\bar{C}_1 + \bar{C}_2\|\omega\|_{C^{0,\gamma}} + \bar{C}_3(\|\mathbf{u}\|_{C^{0,\gamma}} + \|\mathbf{v}\|_{C^{0,\gamma}})}{\epsilon^{2+\alpha(\gamma)}} \|\mathbf{u} - \mathbf{v}\|_{C^{0,\gamma}}, \end{aligned}$$

where we introduce new constants  $\bar{C}_1, \bar{C}_2, \bar{C}_3$ . We let  $\alpha(\gamma) = 0$  if  $\gamma \geq 1/2$ , and  $\alpha(\gamma) = 1/2 - \gamma$  if  $\gamma \leq 1/2$ . One can easily verify that, for all  $\gamma \in (0, 1]$  and  $0 < \epsilon \leq 1$ ,

$$\max \left\{ \frac{1}{\epsilon^2}, \frac{1}{\epsilon^{2+1/2-\gamma}}, \frac{1}{\epsilon^{2-\gamma}} \right\} \leq \frac{1}{\epsilon^{2+\alpha(\gamma)}}.$$

To complete the proof, we combine (88) and (68), and get

$$\|F^\epsilon(y, t) - F^\epsilon(z, t)\|_X \leq \frac{L_1 + L_2(\|\omega\|_{C^{0,\gamma}} + \|y\|_X + \|z\|_X)}{\epsilon^{2+\alpha(\gamma)}} \|y - z\|_X.$$

This proves the Lipschitz continuity of  $F^\epsilon(y, t)$  on any bounded subset of  $X$ . The bound on  $F^\epsilon(y, t)$  (see (12)) follows easily from (75). This completes the proof of Theorem 1.

**4.2. Existence of solution in Hölder space.** In this section, we prove Theorem 2. We begin by proving a local existence theorem. We then show that the local solution can be continued uniquely in time to recover Theorem 2.

The existence and uniqueness of local solutions is stated in the following theorem.

**THEOREM 6** (local existence and uniqueness). *Given  $X = C_0^{0,\gamma}(D; \mathbb{R}^d) \times C_0^{0,\gamma}(D; \mathbb{R}^d)$ , and  $\mathbf{b}(t) \in C_0^{0,\gamma}(D; \mathbb{R}^d)$ , and initial data  $x_0 = (\mathbf{u}_0, \mathbf{v}_0) \in X$ , we suppose that  $\mathbf{b}(t)$  is continuous in time over the interval  $I_0 = (-T, T)$  and satisfies  $\sup_{t \in I_0} \|\mathbf{b}(t)\|_{C^{0,\gamma}(D; \mathbb{R}^d)} < \infty$ . Then, there exist a time interval  $I' = (-T', T') \subset I_0$  and unique solution  $y = (y^1, y^2)$  such that  $y \in C^1(I'; X)$  and*

$$(89) \quad y(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau) d\tau \text{ for } t \in I'$$

or, equivalently,

$$y'(t) = F^\epsilon(y(t), t), \text{ with } y(0) = x_0 \text{ for } t \in I',$$

where  $y(t)$  and  $y'(t)$  are Lipschitz continuous in time for  $t \in I' \subset I_0$ .

To prove Theorem 6, we proceed as follows. We write  $y(t) = (y^1(t), y^2(t))$  and  $\|y\|_X = \|y^1(t)\|_{C^{0,\gamma}} + \|y^2(t)\|_{C^{0,\gamma}}$ . Define the ball  $B(0, R) = \{y \in X : \|y\|_X < R\}$  and choose  $R > \|x_0\|_X$ . Let  $r = R - \|x_0\|_X$ , and we consider the ball  $B(x_0, r)$  defined by

$$(90) \quad B(x_0, r) = \{y \in X : \|y - x_0\|_X < r\} \subset B(0, R);$$

see Figure 7.

To recover the existence and uniqueness we introduce the transformation

$$(91) \quad S_{x_0}(y)(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau) d\tau.$$

Introduce  $0 < T' < T$  and the associated set  $Y(T')$  of Hölder continuous functions taking values in  $B(x_0, r)$  for  $I' = (-T', T') \subset I_0 = (-T, T)$ . The goal is to find the appropriate interval  $I' = (-T', T')$  for which  $S_{x_0}$  maps into the corresponding set  $Y(T')$ . Writing out the transformation with  $y(t) \in Y(T')$  gives

$$(92) \quad S_{x_0}^1(y)(t) = x_0^1 + \int_0^t y^2(\tau) d\tau,$$

$$(93) \quad S_{x_0}^2(y)(t) = x_0^2 + \int_0^t (-\nabla P D^\epsilon(y^1(\tau)) + \mathbf{b}(\tau)) d\tau,$$

and there is a positive constant  $K = C/\epsilon^{2+\alpha(\gamma)}$  (see (12)), independent of  $y^1(t)$ , for  $-T' < t < T'$ , such that the estimation in (93) gives

$$(94) \quad \begin{aligned} & \|S_{x_0}^2(y)(t) - x_0^2\|_{C^{0,\gamma}} \\ & \leq \left( K \left( 1 + \frac{1}{\epsilon^\gamma} + \sup_{t \in (-T', T')} \|y^1(t)\|_{C^{0,\gamma}} \right) + \sup_{t \in (-T, T)} \|\mathbf{b}(t)\|_{C^{0,\gamma}} \right) T', \end{aligned}$$

and from (92)

$$(95) \quad \|S_{x_0}^1(y)(t) - x_0^1\|_{C^{0,\gamma}} \leq \sup_{t \in (-T', T')} \|y^2(t)\|_{C^{0,\gamma}} T'.$$

We write  $b = \sup_{t \in I_0} \|\mathbf{b}(t)\|_{C^{0,\gamma}}$ , and adding (94) and (95) gives the upper bound

$$(96) \quad \|S_{x_0}(y)(t) - x_0\|_X \leq \left( K \left( 1 + \frac{1}{\epsilon^\gamma} + \sup_{t \in (-T', T')} \|y(t)\|_X \right) + b \right) T'.$$

Since  $B(x_0, r) \subset B(0, R)$  (see 90)), we make the choice  $T'$  so that

$$(97) \quad \|S_{x_0}(y)(t) - x_0\|_X \leq \left( \left( K \left( 1 + \frac{1}{\epsilon^\gamma} + R \right) + b \right) \right) T' < r = R - \|x_0\|_X.$$

For this choice we see that

$$(98) \quad T' < \theta(R) = \frac{R - \|x_0\|_X}{K(R + 1 + \frac{1}{\epsilon^\gamma}) + b}.$$

Now it is easily seen that  $\theta(R)$  is increasing with  $R > 0$  and

$$(99) \quad \lim_{R \rightarrow \infty} \theta(R) = \frac{1}{K}.$$

So given  $R$  and  $\|x_0\|_X$  we choose  $T'$  according to

$$(100) \quad \frac{\theta(R)}{2} < T' < \theta(R),$$

and set  $I' = (-T', T')$ . We have found the appropriate time domain  $I'$  such that the transformation  $S_{x_0}(y)(t)$  as defined in (91) maps  $Y(I')$  into itself. We now proceed using standard arguments (see, e.g., [11, Theorem 6.10]) to complete the proof of existence and uniqueness of solution for given initial data  $x_0$  over the interval  $I' = (-T', T')$ .

We now prove Theorem 2. From the proof of Theorem 6 above, we see that a unique local solution exists over a time domain  $(-T', T')$  with  $\frac{\theta(R)}{2} < T'$ . Since  $\theta(R) \nearrow 1/K$  as  $R \nearrow \infty$ , we can fix a tolerance  $\eta > 0$  so that  $[(1/2K) - \eta] > 0$ . Then given any initial condition with bounded Hölder norm and  $b = \sup_{t \in [-T, T]} \|\mathbf{b}(t)\|_{C^{0,\gamma}}$ , we can choose  $R$  sufficiently large so that  $\|x_0\|_X < R$  and  $0 < (1/2K) - \eta < T'$ . Thus we can always find local solutions for time intervals  $(-T', T')$  for  $T'$  larger than  $[(1/2K) - \eta] > 0$ . Therefore, we apply the local existence and uniqueness result to uniquely continue local solutions up to an arbitrary time interval  $(-T, T)$ .

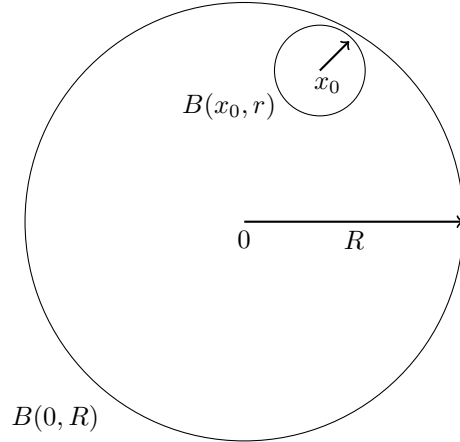


FIG. 7. Geometry.

**5. Limit behavior of Hölder solutions in the limit of vanishing nonlocality.** In this section, we consider the behavior of bounded Hölder continuous solutions as the peridynamic horizon tends to zero. We find that the solutions converge to a limiting sharp fracture evolution with bounded Griffith’s fracture energy and satisfy the linear elastic wave equation away from the fracture set. We look at a subset of Hölder solutions that are differentiable in the spatial variables to show that sharp fracture evolutions can be approached by spatially smooth evolutions in the limit of vanishing nonlocality. As  $\epsilon$  approaches zero, derivatives can become large but must localize to surfaces across which the limiting evolution jumps.

We consider a sequence of peridynamic horizons  $\epsilon_k = 1/k, k = 1, \dots$ , and the associated Hölder continuous solutions  $\mathbf{u}^{\epsilon_k}(t, \mathbf{x})$  of the peridynamic initial value problem (1), (2), and (3). We assume that the initial conditions  $\mathbf{u}_0^{\epsilon_k}, \mathbf{v}_0^{\epsilon_k}$  have uniformly bounded peridynamic energy and mean square initial velocity given by

$$\sup_{\epsilon_k} PD^{\epsilon_k}(\mathbf{u}_0^{\epsilon_k}) < \infty \quad \text{and} \quad \sup_{\epsilon_k} \|\mathbf{v}_0^{\epsilon_k}\|_{L^2(D; \mathbb{R}^d)} < \infty.$$

Moreover, we suppose that  $\mathbf{u}_0^{\epsilon_k}, \mathbf{v}_0^{\epsilon_k}$  are differentiable on  $D$  and that they converge in  $L^2(D; \mathbb{R})$  to  $\mathbf{u}_0^0, \mathbf{v}_0^0$  with bounded Griffith’s free energy given by

$$\int_D 2\mu |\mathcal{E}\mathbf{u}_0^0|^2 + \lambda |\text{div } \mathbf{u}_0^0|^2 dx + \mathcal{G}_c \mathcal{H}^{d-1}(J_{\mathbf{u}_0^0}) \leq C < \infty,$$

where  $J_{\mathbf{u}_0^0}$  denotes an initial fracture surface given by the jumps in the initial deformation  $\mathbf{u}_0^0$  and  $\mathcal{H}^2(J_{\mathbf{u}_0^0(t)})$  is its 2 dimensional Hausdorff measure of the jump set. Here  $\mathcal{E}\mathbf{u}_0^0$  is the elastic strain and  $\text{div } \mathbf{u}_0^0 = \text{Tr}(\mathcal{E}\mathbf{u}_0^0)$ . The constants  $\mu, \lambda$  are given by the explicit formulas

$$\mu = \lambda = \frac{1}{5} f'(0) \int_0^1 r^d J(r) dr, \quad d = 2, 3,$$

and

$$\mathcal{G}_c = \frac{3}{2} f_\infty \int_0^1 r^d J(r) dr, \quad d = 2, 3,$$



where  $f'(0)$  and  $f_\infty$  are defined by (7). Here  $\mu = \lambda$  and is a consequence of the central force model used in cohesive dynamics. Last we suppose as in [23] that the solutions are uniformly bounded, i.e.,

$$\sup_{\epsilon_k} \sup_{[0, T]} \|\mathbf{u}^{\epsilon_k}(t)\|_{L^\infty(D; \mathbb{R}^d)} < \infty.$$

The Hölder solutions  $\mathbf{u}^{\epsilon_k}(t, \mathbf{x})$  naturally belong to  $L^2(D; \mathbb{R}^d)$  for all  $t \in [0, T]$  and we can directly apply the Gronwall inequality (equation (6.9) of [23]) together with Theorems 6.2 and 6.4 of [23] to conclude similarly to Theorems 5.1 and 5.2 of [23] that there are at least one “cluster point”  $\mathbf{u}^0(t, \mathbf{x})$  belonging to  $C([0, T]; L^2(D; \mathbb{R}^d))$  and a subsequence, also denoted by  $\mathbf{u}^{\epsilon_k}(t, \mathbf{x})$ , for which

$$\lim_{\epsilon_k \rightarrow 0} \max_{0 \leq t \leq T} \{\|\mathbf{u}^{\epsilon_k}(t) - \mathbf{u}^0(t)\|_{L^2(D; \mathbb{R}^d)}\} = 0.$$

Moreover, it follows from [23] that the limit evolution  $\mathbf{u}^0(t, \mathbf{x})$  has a weak derivative  $\mathbf{u}_t^0(t, \mathbf{x})$  belonging to  $L^2([0, T] \times D; \mathbb{R}^d)$ . For each time  $t \in [0, T]$  we can apply methods outlined in [23] to find that the cluster point  $\mathbf{u}^0(t, \mathbf{x})$  is a special function of bounded deformation (see [4] and [5]) and has bounded linear elastic fracture energy given by

$$\int_D 2\mu |\mathcal{E}\mathbf{u}^0(t)|^2 + \lambda |\operatorname{div} \mathbf{u}^0(t)|^2 dx + \mathcal{G}_c \mathcal{H}^2(J_{\mathbf{u}^0(t)}) \leq C$$

for  $0 \leq t \leq T$ , where  $J_{\mathbf{u}^0(t)}$  denotes the evolving fracture surface. The deformation crack set pair  $(\mathbf{u}^0(t), J_{\mathbf{u}^0(t)})$  records the brittle fracture evolution of the limit dynamics.

Arguments identical to [23] show that away from sets where  $|S(\mathbf{y}, \mathbf{x}; \mathbf{u}^{\epsilon_k})| > S_c$  the limit  $\mathbf{u}^0$  satisfies the linear elastic wave equation. This is stated as follows: Fix  $\delta > 0$ , and for  $\epsilon_k < \delta$  and  $0 \leq t \leq T$  consider the open set  $D' \subset D$  for which points  $\mathbf{x}$  in  $D'$  and  $\mathbf{y}$  for which  $|\mathbf{y} - \mathbf{x}| < \epsilon_k$  satisfy

$$|S(\mathbf{y}, \mathbf{x}; \mathbf{u}^{\epsilon_k}(t))| < S_c(\mathbf{y}, \mathbf{x}).$$

Then the limit evolution  $\mathbf{u}^0(t, \mathbf{x})$  evolves elastodynamically on  $D'$  and is governed by the balance of linear momentum expressed by the Navier–Lamé equations on the domain  $[0, T] \times D'$  given by

$$\mathbf{u}_{tt}^0(t) = \operatorname{div} \boldsymbol{\sigma}(t) + \mathbf{b}(t) \quad \text{on } [0, T] \times D',$$

where the stress tensor  $\boldsymbol{\sigma}$  is given by

$$\boldsymbol{\sigma} = \lambda I_d \operatorname{Tr}(\mathcal{E} \mathbf{u}^0) + 2\mu \mathcal{E} \mathbf{u}^0,$$

where  $I_d$  is the identity on  $\mathbb{R}^d$  and  $\operatorname{Tr}(\mathcal{E} \mathbf{u}^0)$  is the trace of the strain. Here the second derivative  $\mathbf{u}_{tt}^0$  is the time derivative in the sense of distributions of  $\mathbf{u}_t^0$ , and  $\operatorname{div} \boldsymbol{\sigma}$  is the divergence of the stress tensor  $\boldsymbol{\sigma}$  in the distributional sense. This shows that sharp fracture evolutions can be approached by spatially smooth evolutions in the limit of vanishing nonlocality.

**6. Conclusions.** In this article, we have presented a numerical analysis for a class of nonlinear nonlocal peridynamic models. We have shown that the convergence rate applies, even when the fields do not have well-defined spatial derivatives. We treat the forward Euler scheme as well as the general implicit single step method.

The convergence rate is found to be the same for both schemes and is given by  $C(\Delta t + h^\gamma/\epsilon^2)$ . Here the constant  $C$  depends on  $\epsilon$  and Hölder and the  $L^2$  norm of the solution and its time derivatives. The Lipschitz property of the nonlocal, nonlinear force together with boundedness of the nonlocal kernel plays an important role. It ensures that the error in the nonlocal force remains bounded when replacing the exact solution with its approximation. This, in turn, implies that even in the presence of mechanical instabilities the global approximation error remains controlled by the local truncation error in space and time.

Taking  $\gamma = 1$ , a straightforward estimate of (38) using (12) gives to leading order

$$(101) \quad \sup_{0 \leq k \leq T/\Delta t} E^k \leq \left[ C_1 \Delta t C_t + C_2 h \sup_{0 < t < T} \|u\|_{C^{0,1}(D; \mathbb{R}^3)} \right],$$

where  $C_t$  is independent of  $\epsilon$  and depends explicitly on the  $L^2$  norms of time derivatives of the solution; see (37) and

$$C_1 = \exp [T(1 + 6\bar{C}/\epsilon^2)] T,$$

$$C_2 = \exp [T(1 + 6\bar{C}/\epsilon^2)] T \left( 1 + \sqrt{3}\bar{C} \left( 1 + \frac{1}{\epsilon} \right) + \frac{4\sqrt{3}\bar{C}}{\epsilon^2} \right).$$

It is evident that the exponential factor could be large. However, we can choose times  $T$  for which the effects of the exponential factor can be diminished and  $C_1$  and  $C_2$  are not too large. To fix these ideas, we consider a 1 cubic meter sample and a corresponding 1400 meter per second shear wave speed. This wave speed is characteristic of plexiglass. Then the time for a shear wave to traverse the sample is 718  $\mu$ -seconds. This is the characteristic time  $T^*$  and a fracture experiment can last a few hundred  $\mu$ -seconds. The actual time in  $\mu$ -seconds of a fracture simulation is given by  $TT^*$  where  $T$  is the nondimensional simulation time. The dimensionless constant  $\bar{C}$  is 1.19, and we take  $\epsilon = 1/10$  and dimensionless body force unity. For a simulation cycle of length  $TT^* = 1.5\mu$ -seconds the constants  $C_1$  and  $C_2$  in (101) are 0.0193 and 7.976, respectively. The solution after cycle time  $T$  can be used as initial conditions for a subsequent run and the process can be iterated. Unfortunately, these estimates predict a total simulation time of 15 $\mu$ -second before the relative error becomes greater than one even for a large number of spatial degrees of freedom. We point out that because the constants in the a priori bound are necessarily pessimistic, the predicted simulation time is an order of magnitude below what is seen in the experiment. Future work will focus on a posteriori estimates for simulations and adaptive implementations of the finite difference scheme.

In conclusion, the analysis shows that the method is stable, and one can control the error by choosing the time step and spatial discretization sufficiently small. However, errors do accumulate with time steps and this limits the time interval for simulation. We have identified local perturbations for which the error accumulates with the time step for the implicit Euler method. These unstable local perturbations correspond to regions for which a preponderance of bonds are in the softening regime.

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