Numerical fracture experiments using nonlinear nonlocal models

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Joint work with
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Overview of the talk

- Nonlinear peridynamic model
- Well-posedness of nonlocal model
- Numerical results
- Ongoing and future work
Nonlinear peridynamic model

Let $\epsilon$ be the horizon, $B_\epsilon(x)$ ball of radius $\epsilon$, and $u(x)$ displacement of material point $x \in D$. In this work, we consider linearized pairwise strain $S(y, x; u)$ given by

$$S(y, x; u) = \frac{u(y) - u(x)}{|y - x|} \cdot \frac{y - x}{|y - x|}.$$

Suppose $\hat{f}^\epsilon(y, x)$ denotes the force applied on $x$ from the neighboring point $y$. Then total force at $x$ is given by

$$f^\epsilon(x) = \int_{B_\epsilon(x)} \hat{f}^\epsilon(y, x) dy.$$

We consider pairwise force based on smooth and concave potential function $\psi^{1,2}$

$$\hat{f}^\epsilon(y, x) = \frac{1}{\epsilon |B_\epsilon(0)|} \frac{\partial S}{\partial S \psi(|y - x|S(y, x)^2)} \frac{y - x}{|y - x|}.$$


Equation of motion

\[ \rho \ddot{u}(x, t) = f^e(x; u(t)) + b(x, t), \quad \forall x \in D, t \in [0, T] \]

Boundary condition

\[ u(x, t) = g(x, t) \quad \forall x \in D_u, t \in [0, T] \]

\[ b(x, t) = f_{\text{ext}}(x, t) \quad \forall x \in D_f, t \in [0, T] \]

\( D_u, D_f \subset D \) are layers with finite volume (area in 2-d) on which displacement and external force, respectively, are specified. External force is applied in the form of body force.

Initial condition: \( u(x, 0) = u_0(x), \dot{u}(x, 0) = v_0(x) \) for all \( x \in D \).
Well-posedness of nonlinear peridynamic model

- Using the fact that nonlinear peridynamic force is bounded and Lipschitz continuous with respect to displacement field $\mathbf{u} \in L^2_0(D)$, the existence of solutions over any finite time domain $[0, T]$ is shown [1].

- To prove existence of solutions in more regular spaces, we introduce boundary function $\omega$ into Peridynamic force. $\omega(x) = 1$ in the interior and smoothly decays to 0 as $x$ approaches boundary $\partial D$.

- To perform apriori error analysis of finite difference approximation, we consider Hölder space $C_0^{0, \gamma}(D), \gamma \in (0, 1]$. In [2] we show existence of solutions in Hölder space $C_0^{0, \gamma}(D)$. In [3] we extend the results to state-based peridynamic models.

- For apriori error analysis of finite element approximation using continuous piecewise linear elements, we consider natural space $H^2(D) \cap H^1_0(D)$. In [4] we show existence of solutions in $H^2(D) \cap H^1_0(D)$.

Finite difference approximation

We approximate peridynamic equation using **piecewise constant interpolation** and central in time discretization. Let $\mathbf{u}_i^k$ denote the discrete displacement at mesh node $\mathbf{x}_i$ and time $t^k = k\Delta t$. We consider the following piecewise constant function

$$
\mathbf{u}_h^k(x) = \sum_{i, \mathbf{x}_i \in D} \mathbf{u}_i^k \chi_{U_i}(x)
$$

Discrete problem is

$$
\frac{\mathbf{u}_h^{k+1} - 2\mathbf{u}_h^k + \mathbf{u}_h^{k-1}}{\Delta t^2} = \mathbf{f}_h^\varepsilon(t^k) + \mathbf{b}_h^k,
$$

where

$$
\mathbf{f}_h^\varepsilon(x, t^k) = \sum_{i, \mathbf{x}_i \in D} \mathbf{f}^\varepsilon(\mathbf{x}_i, t^k) \chi_{U_i}(x),
$$

$$
\mathbf{b}_h(x, t^k) = \sum_{i, \mathbf{x}_i \in D} \mathbf{b}(\mathbf{x}_i, t^k) \chi_{U_i}(x).
$$
Convergence of finite difference approximation

Error at time step $k$ is defined as: $E^k = ||u_h^k - u(t^k)||$.

**Theorem 1.** Let $\epsilon > 0$ be fixed. Let $(u, v)$ be the solution of peridynamic equation. We assume $u, v \in C^2([0, T]; C^{0,\gamma}(D; \mathbb{R}^d))$. Then the finite difference scheme is consistent in both time and spatial discretization and converges to the exact solution uniformly in time with respect to the $L^2$ norm. If we assume the error at the initial step is zero then the error $E^k$ at time $t^k$ is bounded and satisfies

$$\sup_{0 \leq k \leq T/\Delta t} E^k \leq O \left( C_t \Delta t + C_s \frac{h^\gamma}{\epsilon^2} \right),$$

where constant $C_s$ and $C_t$ are independent of $h$ and $\Delta t$. Constants $C_t, C_s$ depend on the $\epsilon$ and Hölder norm of the exact solution.

Setting up peridynamic model

- Pairwise potential: \( \psi(r) = c(1 - \exp[-\beta r^2]) \)
- Influence function: \( J(r) = 1 - r \) for \( 0 \leq r < 1 \) and \( J(r) = 0 \) for \( r \geq 1 \)
- Critical strain: \( S_c(y, x) = \frac{\pm \bar{r}}{\sqrt{|y - x|}} \), where \( \bar{r} \) is the inflection point of function \( \psi \)
- We fix \( \rho = 1200 \text{ kg/m}^3 \), bulk modulus \( K = 3.24 \text{ GPa} \), critical energy release rate \( G_c = 500 \text{ J/m}^{-2} \)
- Using relation between nonlinear peridynamic model and linear elastic fracture mechanics\(^1\), we find
  \[
  c = 4712.4, \quad \beta = 1.7533 \times 10^8, \quad \bar{r} = \frac{1}{\sqrt{2\beta}} = 5.3402 \times 10^{-5}
  \]

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Mode I crack propagation: Setup

Goal: Localization of crack and convergence to classical fracture mechanics for simple mode-I crack propagation\(^1\)

- Final time \(T = 560 \mu s\), time step \(\Delta t = 0.02 \mu s\)
- Uniform grid on square domain \(D = [0, 0.1 \text{ m}] \times [-0.15 \text{ m}, 0.15 \text{ m}]\)
- Experiment with three different horizons \(\epsilon = 2.5, 1.25, 0.625 \text{ mm}\)
- Body force \(b^\epsilon(x, t) = (0, f_0 h(t)/\epsilon)\) on top layer and \(b^\epsilon(x, t) = (0, -f_0 h(t)/\epsilon)\) on bottom layer
- \(h(t)\) is a step function such that \(h(t) = t\) for \(t \leq 350 \mu s\) and \(h(t) = 1\) for \(t > 350 \mu s\)
- Mesh size is fixed by relation \(h = \epsilon/4\)

Localization of fracture zone

$t = 460 \mu s$

(a) (b) (c)

$t = 520 \mu s$

(d) (e) (f)
Localization of fracture zone

\[ t = 520 \mu s \]
Crack tip location and velocity

- Crack tip location as a function of time for different horizon lengths.
- Velocity profile of the crack tip over time for different horizon lengths.

- $c_R = 969.08 \text{ m/s}$
Energy into crack

The energy associated to crack is given by\(^1\)

\[
E(\Gamma_\delta(t)) = \frac{1}{|B_\epsilon(0)|} \int_{P_\delta(t)} \int_{P_\delta(t) \cap B_\epsilon(x)} \partial_S W(S(y, x; u(t))) \frac{y - x}{|y - x|} \cdot (\dot{u}(x, t) + \ddot{u}(y, t)) dy dx.
\]

Here \(W\) is the peridynamic pairwise energy density. \(P_\delta(t)\) is the rectangle domain with crack tip at its center. It is moving with tip. \(P_\delta^c(t)\) is the complement of \(P_\delta(t)\).

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Energy into crack

![Graph showing energy into crack over time. The graph plots energy (J/(m x s)) against time (μs) with two lines: $E^\varepsilon(\Gamma^\varepsilon(t))$ and $V(t)G_c$.](image-url)
Mix mode crack propagation

Material properties are same as in the Mode-I problem. We set

- Horizon $\epsilon = 0.5$ mm
- Mesh size $h = 0.125$ mm
- Final time $T = 140 \mu$s
- Time step size $\Delta t = 0.004 \mu$s


Crack-void interaction

Material properties are same as in the Mode-I problem. We set

- Horizon $\epsilon = 0.4$ mm
- Mesh size $h = 0.1$ mm
- Final time $T = 800$ $\mu$s
- Time step size $\Delta t = 0.004$ $\mu$s

(b) Damage profile

(c) Magnitude of symmetric gradient of displacement

(d) Numerical experiment results using FEM, Boundary element method [2]


Wave reflection effect on crack velocity

We consider a softer material with shear modulus $G = 35.2 \text{ kPa}$, density $\rho = 1011 \text{ kg/m}^3$, and critical energy release rate $G_c = 20 \text{ J/m}^{-2}$. Poisson ratio is fixed to $\mu = 0.25$. Domain is $D = [0, 0.12 \text{ m}] \times [0, 0.03 \text{ m}]$.

- Horizon $\epsilon = 0.6 \text{ mm}$, mesh size $h = 0.15 \text{ mm}$
- Time $T = 1.1 \text{ s}$, $\Delta t = 2.2 \mu \text{s}$
Wave reflection effect on crack velocity

- Max crack length = 0.12 m
- Rayleigh wave speed $c_R = 5.502 \text{ m/s}$
Ongoing and future works

- In [1] we show that the classical kinetic relation is embedded in peridynamics and we have \( \lim_{\epsilon \to 0} J(t) = G_c \), where \( J(t) \) is the nonlocal J-integral. In LEFM, the classical kinetic relation for the crack velocity is postulated. In contrast, we obtain the classical kinetic relation from the Peridynamics in the limit of vanishing nonlocality.

- Open source computational library for nonlocal modeling. This is a joint work with Patrick Diehl (LSU) and Robert Lipton (LSU).

- Study of granular material using nonlinear nonlocal model.

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Thank you!