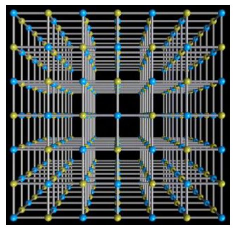
Numerical Analysis of Nonlocal Fracture Models



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Research Talk

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University of Minnesota, Minneapolis

*Figure : http://www.themolecularuniverse.com/mile/mile1.htm

Overview of the talk

- Brief Introduction to Peridynamics
- Nonlinear potential in our model
- Numerical analysis in Holder space
- Numerical analysis in one dimension
- Numerical verification
- Wave dispersion
- Discussion and future works

Peridynamics theory

Let D be the material domain, D_c be nonlocal boundary, and $\boldsymbol{u}$ be the displacement field.

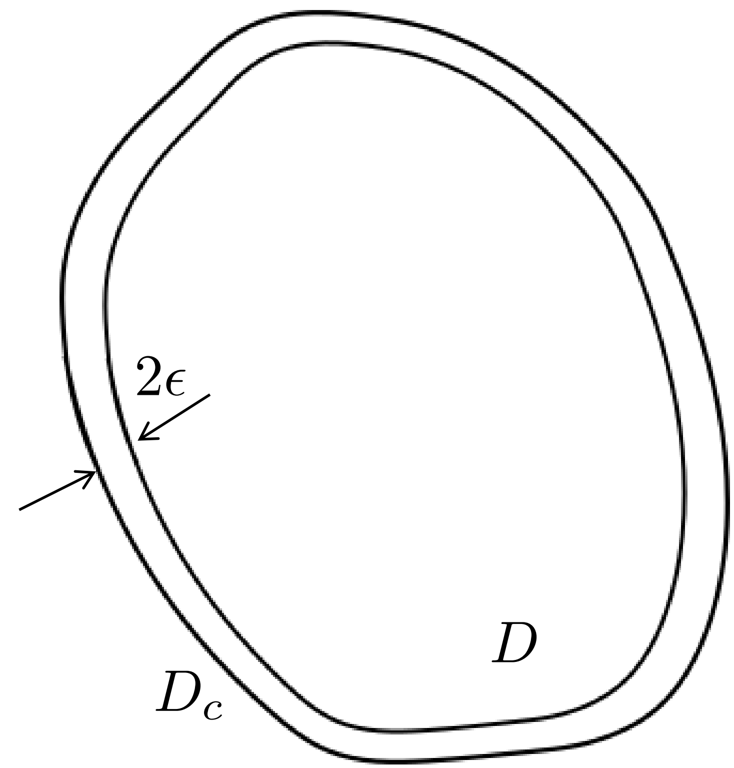
Let $\boldsymbol{x}$ denote the material point and $\boldsymbol{\chi}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ is the deformed position.

Strain between two material point $\boldsymbol{x}$ and $\boldsymbol{y}$ is given by

$$S(\boldsymbol{y}, \boldsymbol{x}; \boldsymbol{u}) = \frac{\boldsymbol{y} + \boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{x} - \boldsymbol{u}(\boldsymbol{x})}{|\boldsymbol{y} - \boldsymbol{x}|} \cdot \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|}$$

Assuming that displacement is small compared to the size of material, we linearize S and get

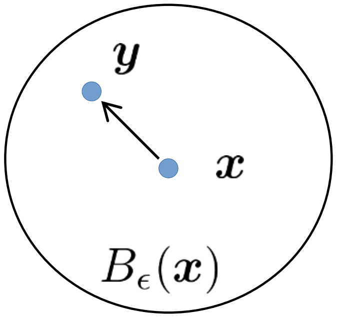
$$S(\boldsymbol{y}, \boldsymbol{x}; \boldsymbol{u}) = \frac{\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})}{|\boldsymbol{y} - \boldsymbol{x}|} \cdot \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|}$$



Peridynamics theory...

Consider a material point $\boldsymbol{x}$. We introduce a length scale ϵ which is called size of horizon. This controls the extent of nonlocal interaction in the material.

Generic form of force at $\boldsymbol{x}$ in peridynamic model is given by



$$\boldsymbol{f}^\epsilon(\boldsymbol{x};\boldsymbol{u}) = \frac{2}{\omega_d \epsilon^d} \int_{B_\epsilon(\boldsymbol{x})} \boldsymbol{f}^\epsilon(\boldsymbol{y},\boldsymbol{x};\boldsymbol{u}) d\boldsymbol{y}$$

$\boldsymbol{f}^\epsilon$ depends on choice of ϵ.

In the limit, $\epsilon \to 0$ the model should collapse to classical mechanics.

Given ϵ, we fit the parameters in $\boldsymbol{f}^\epsilon$, so that fracture toughness G and Poissons ratio μ remains smae.

Nonlocal nonlinear potential

We consider following type of nonlocal potential:

$$W^\epsilon(\boldsymbol{y},\boldsymbol{x};S) = \frac{1}{\epsilon} J\left(\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\epsilon}\right)\psi(|\boldsymbol{y}-\boldsymbol{x}|S^2)$$

The force at $\boldsymbol{x}$ is given by

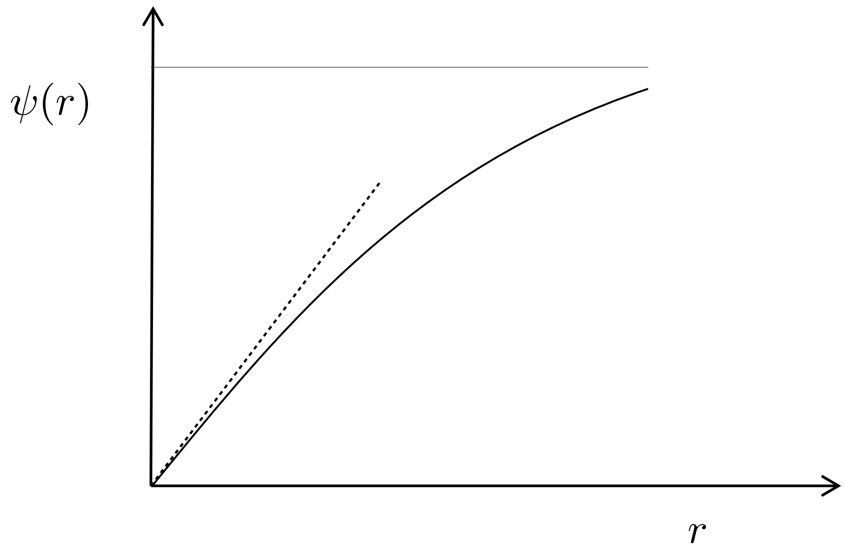
$$\boldsymbol{f}^\epsilon(\boldsymbol{x},\boldsymbol{u}) = \frac{2}{\omega_d \epsilon^d}\int_{B_\epsilon(\boldsymbol{x})} \partial_S W^\epsilon(\boldsymbol{y},\boldsymbol{x};S)\frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|}d\boldsymbol{y}$$

$J(r)$ is the influence function. Controls the effect of bond $|\boldsymbol{y}-\boldsymbol{x}|$ on force at $\boldsymbol{x}$.

This form of potential is introduced and analysed in detail in **Lipton 2016** *Cohesive dynamics and brittle fracture*.

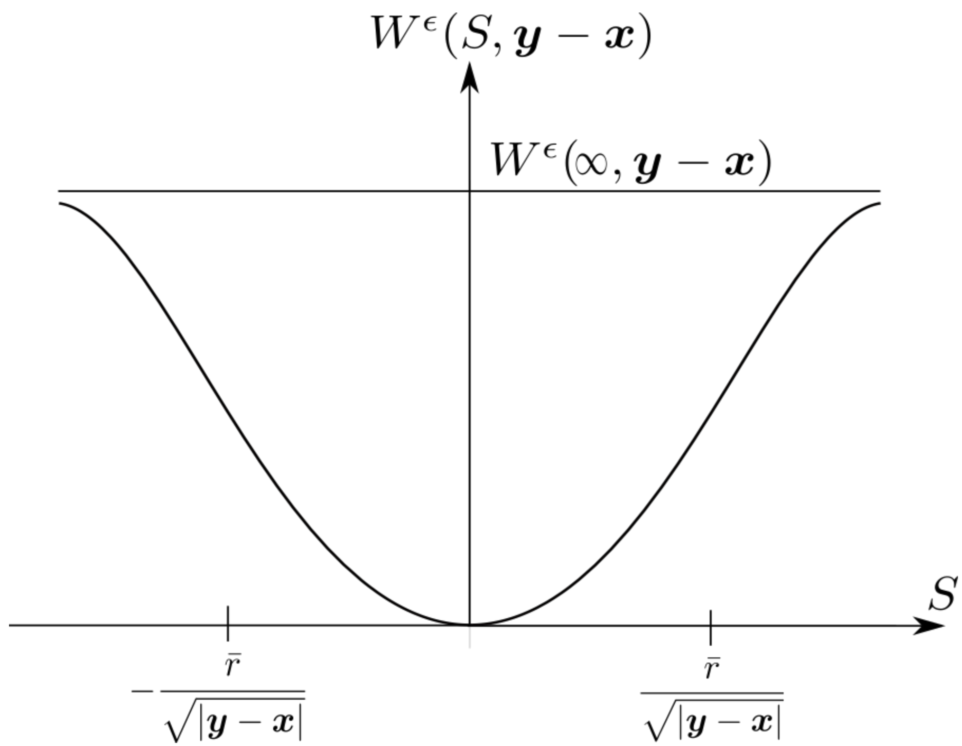
$\psi(r)$ is the nonlinear potential. We assumed it to be smooth, positive, and concave and satisfies following properties

$$\lim_{r\to 0^+} \frac{\psi(r)}{r} = \psi'(0) \text{ and } \lim_{r\to\infty}\psi(r) = \psi_\infty < \infty$$

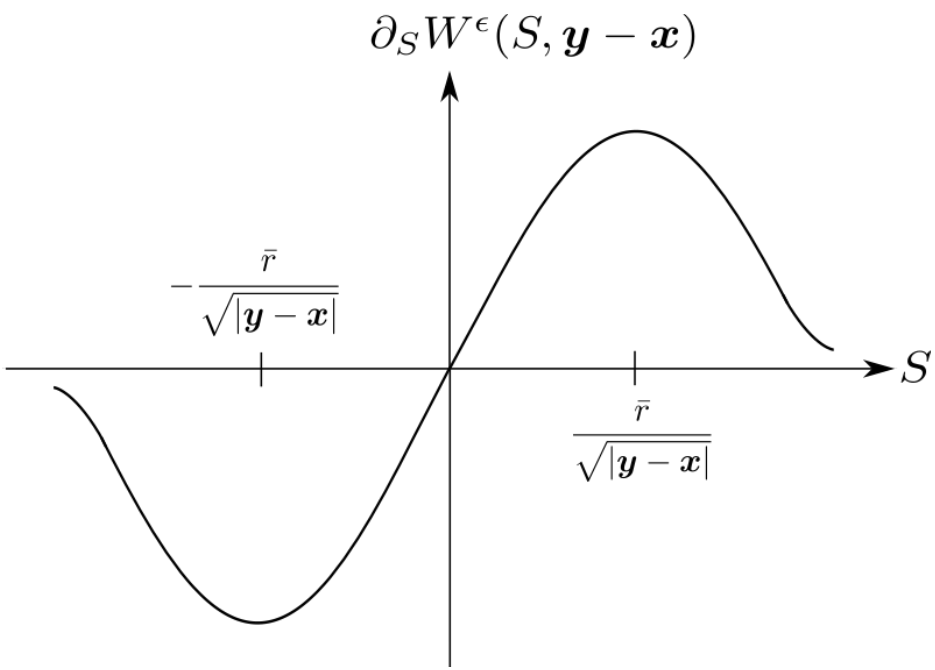


Nonlocal potential

$W^\epsilon(S)$ has two wells: at $r = 0$ corresponding to linear elasticity and at $r = \infty$ corresponding to fracture.



(a) Strain vs Potential



(b) Strain vs Force

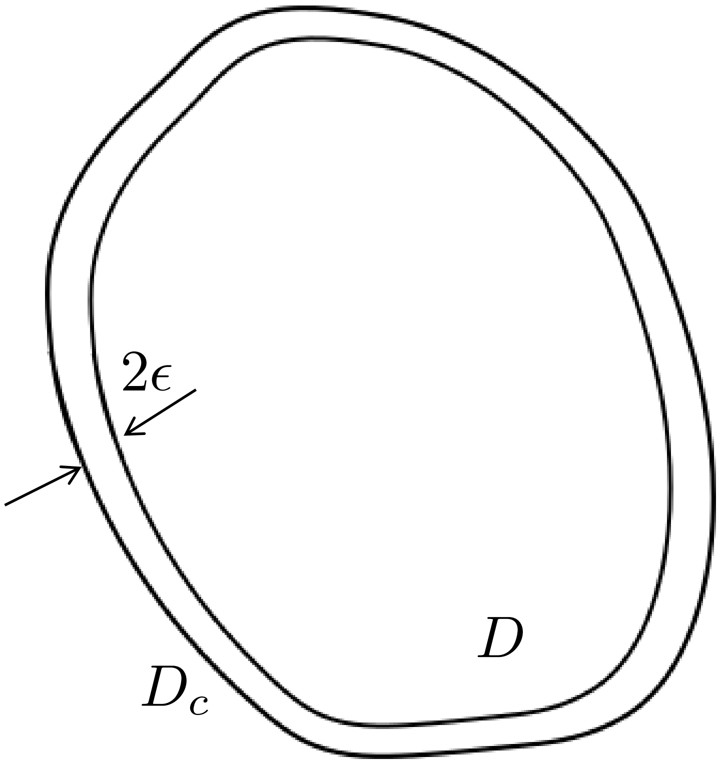
Boundary condition and initial condition

- We consider nonlocal boundary condition in layer of thickness 2ϵ denoted by D_c. We consider following b.c.

$$\boldsymbol{u} = \boldsymbol{0} \qquad \forall \boldsymbol{x} \in D_c$$

- Initial condition is given by

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \text{ and } \dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}) \quad \boldsymbol{x} \in D$$



Theorem 6.1 in **Lipton 2016** gives the existence of solution $\boldsymbol{u}^\epsilon \in C^2([0,T]; L^2(D; \mathbb{R}^d))$ satisfying above boundary condition and initial condition with $\boldsymbol{u}_0.\boldsymbol{v}_0 \in L^2(D; \mathbb{R}^d)$.

References

- The peridynamics model was introduced by **Silling 2000** in *Reformulation of elasticity theory for discontinuities and long-range forces.*

- There has been extensive amount of work published in application of peridynamics for crack propagation. Few of them are as follows:

  Silling, Weckner, Askari, and Bobaru 2010 *Crack nucleation in a peridynamic solid*, International Journal of Fracture.

  Bobaru, Florin, Hu, and Wenke *Studies of dynamic crack propagation and crack branching with peridynamics.*

  Ha, Youn, Bobaru, and Florin *Studies of dynamic crack propagation and crack branching with peridynamics.*

  Agwai, Abigail ... *Predicting crack propagation with peridynamics: a comparative study.*

  Lipton 2016 *Cohesive dynamics and brittle fracture.*

  Lipton, Silling, and Lehoucq *Complex fracture nucleation and evolution with nonlocal elastodynamics.*

  Lipton 2014 *Dynamic brittle fracture as a small horizon limit of peridynamics.*

- Limit of peridynamic equation:

  Silling and Lehoucq 2008 *Convergence of peridynamics to classical elasticity theory.*

  Lipton 2016 *Cohesive dynamics and brittle fracture.*

References...

Numerical analysis of Peridynamic model

Silling and Askari 2005 *A meshfree method based on the peridynamic model of solid mechanics.*

Weckner and Emmrich *Numerical simulation of the dynamics of a nonlocal, inhomogeneous, infinite bar.*

Bobaru, Yang ... 2009 *Convergence, adaptive refinement, and scaling in 1D peridynamics.*

Chen and Gunzburger 2011 *Continuous and discontinuous finite element methods for a peridynamics model of mechanics.*

Du, Gunzburger... 2012 *Analysis and approximation of nonlocal diffusion problems with volume constraints.*

Mengesha and Du 2013 *Analysis of a scalar peridynamic model with a sign changing kernel.*

Tain, Du, and Gunzburger 2016 *Asymptotically compatible schemes for the approximation of fractional Laplacian and related nonlocal diffusion problems on bounded domains.*

Diehl, Lipton, and Schiweitzer 2016 *Numerical verification of a bond-based softening peridynamic model for small displacements: Deducing material parameters from classical linear theory.*

Jha and Lipton 2017 *Numerical analysis of nonlocal fracture models in Hölder space.*

Numerical analysis in Holder space

Let $\gamma \in (0,1]$ be Hölder exponent and let $C^{0,\gamma}(D;\mathbb{R}^d)$ be the Hölder space.

- Existence of a solution
- Convergence and stability

Existence of solution

To show existence, we first extend following result in **Lipton 2016** which shows the Lipschitz continuity of $\boldsymbol{f}^\epsilon$

$$||\boldsymbol{f}^\epsilon(\cdot;\boldsymbol{u}) - \boldsymbol{f}^\epsilon(\cdot;\boldsymbol{v})||_{L^2(D;\mathbb{R}^d)} \leq \frac{L}{\epsilon^2}||\boldsymbol{u}-\boldsymbol{v}||_{L^2(D;\mathbb{R}^d)}$$

Proposition 1 *For any $\boldsymbol{u},\boldsymbol{v} \in C^{0,\gamma}(D;\mathbb{R}^d)$, we have*

$$||\boldsymbol{f}^\epsilon(\cdot;\boldsymbol{u}) - \boldsymbol{f}^\epsilon(\cdot;\boldsymbol{v})||_{C^{0,\gamma}} \leq \frac{L_1 + L_2(||\boldsymbol{u}||_{C^{0,\gamma}} + ||\boldsymbol{v}||_{C^{0,\gamma}})}{\epsilon^{2+\alpha(\gamma)}}||\boldsymbol{u}-\boldsymbol{v}||_{C^{0,\gamma}}$$

Existence of solution in Holder space

Let $X := C_0^{0,\gamma}(D;\mathbb{R}^d) \times C_0^{0,\gamma}(D;\mathbb{R}^d)$ and let $y = (y_1, y_2) \in X$. Letting $y_1 = \boldsymbol{u}$ and $y_2 = \dot{\boldsymbol{u}}$. Define $F^\epsilon(y,t) = (F_1^\epsilon(y,t), F_2^\epsilon(y,t))$

$$\begin{aligned} F_1^\epsilon(y,t) &:= y_2 \\ F_2^\epsilon(y,t) &:= \boldsymbol{f}^\epsilon(y_1) + \boldsymbol{b}(t). \end{aligned}$$

Then first order equivalent system is

$$\frac{d}{dt}y(t) = F^\epsilon(y,t) \quad \forall t \in J := [-T,T]$$

with initial condition $y(0) = x_0 = (\boldsymbol{u}_0, \boldsymbol{v}_0)$.

To prove the existence of solution, we need to show that $y(t)$

$$y(t) := x_0 + \int_0^t F^\epsilon(\tau, y(\tau))d\tau$$

exists in X for all $t \in J$.

Existence of solution in Holder space...

We show existence in two steps:

Local existence: $\exists$ time domain $J' = (-T', T')$ such that $y(t)$ is in $C^{0,\gamma}$ for all $t \in J'$.

Global existence: Given time domain $J = [-T, T]$, we construct solution on J by keep on using local existence theorem for every small time domain J'. For this, we need T' such that we can choose it independent of initial condition x_0. We show that this is possible in our problem.

Theorem 2 *For any initial condition $x_0 \in X$, interval $J = (-T, T)$, and right hand side $\boldsymbol{b}(t)$ continuous in time for $t \in J_0$ and $\sup_{t \in J_0} ||\boldsymbol{b}(t)||_{C^{0,\gamma}} < \infty$, there is a unique solution $y(t) \in C^1(J; X)$ of*

$$y(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau)\, d\tau,$$

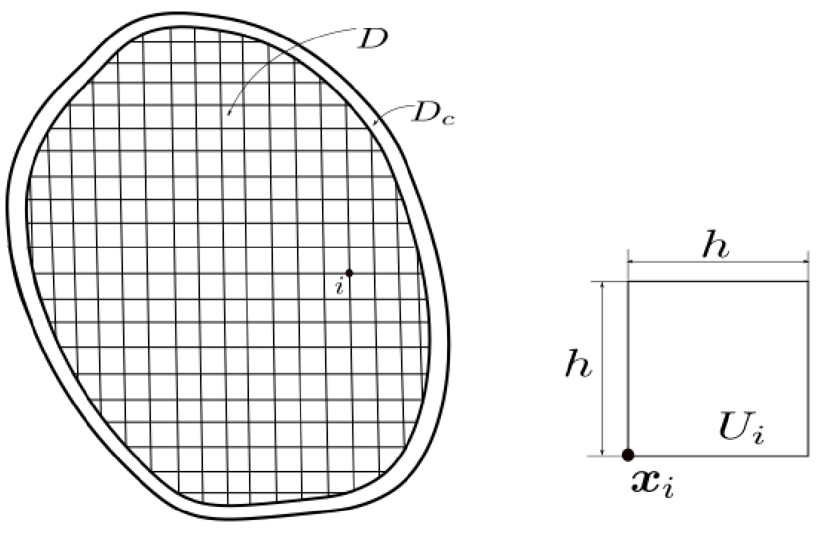
or equivalently

$$\frac{d}{dt} y(t) = F^\epsilon(y(t), t), \textit{with } y(0) = x_0,$$

where $y(t)$ and $y'(t)$ are Lipschitz continuous in time for $t \in J$.

Finite difference approximation

We consider Forward Euler scheme to integrate the peridynamic equation in time. However, the results extend to implicit scheme like Crank Nicholson and Backward Euler very easily and have similar stability and convergence property.



(a) (b)

Let $D_h = D \cap h\mathbb{Z}^d$ be the mesh corresponding to mesh size h. Let $\Delta t > 0$ be the time step.

Denote $x_i = ih$ with $i \in \mathbb{Z}^d$ and $t^k = k\Delta t$ with $0 \leq k \leq T/\Delta t$.

Let $(\hat{\boldsymbol{u}}_i^k, \hat{\boldsymbol{v}}_i^k)$ be discrete solution at $t = t^k$ and $x = x_i$. This satisfies

$$\frac{\hat{\boldsymbol{u}}_i^{k+1} - \hat{\boldsymbol{u}}_i^k}{\Delta t} = \hat{\boldsymbol{v}}_i^k$$
$$\frac{\hat{\boldsymbol{v}}_i^{k+1} - \hat{\boldsymbol{v}}_i^k}{\Delta t} = \boldsymbol{f}^\epsilon(\hat{\boldsymbol{u}}^k)(\boldsymbol{x}_i) + \boldsymbol{b}_i^k$$

We denote piecewise constant extension of discrete solution as $\hat{\boldsymbol{u}}^k$ and $\hat{\boldsymbol{v}}^k$ (without indices i).

$$\hat{\boldsymbol{u}}^k(\boldsymbol{x}) := \sum_{i, \boldsymbol{x}_i \in D} \hat{\boldsymbol{u}}_i^k \chi_{U_i}(\boldsymbol{x})$$
$$\hat{\boldsymbol{v}}^k(\boldsymbol{x}) := \sum_{i, \boldsymbol{x}_i \in D} \hat{\boldsymbol{v}}_i^k \chi_{U_i}(\boldsymbol{x})$$

Finite difference approximation...

Given $(\boldsymbol{u}, \boldsymbol{v})$ as exact solution, we define the error E^k at time step k as follows

$$E^k := ||\hat{\boldsymbol{u}}^k - \boldsymbol{u}^k||_{L^2} + ||\hat{\boldsymbol{v}}^k - \boldsymbol{v}^k||_{L^2}$$

where $\boldsymbol{u}^k = \boldsymbol{u}(t^k)$ and $\boldsymbol{v}^k = \boldsymbol{v}(t^k)$.

Theorem 3 *Let $\epsilon > 0$ be fixed. Let $(\boldsymbol{u}, \boldsymbol{v})$ be the solution of peridynamic equation. We assume $\boldsymbol{u}, \boldsymbol{v} \in C^2([0,T]; C^{0,\gamma}(D; \mathbb{R}^d))$. Then the finite difference scheme is consistent in both time and spatial discretization and converges to the exact solution uniformly in time with respect to the $L^2(D;\mathbb{R}^d)$ norm. We assume the error at the initial step is zero then the error E^k at time t^k is bounded and satisfies*

$$\sup_{0 \le k \le T/\Delta t} E^k = O\left(\Delta t + \frac{h^\gamma}{\epsilon^2}\right).$$

Sketch of Proof: We divide the error E^k in two parts as follows

$$||\hat{\boldsymbol{u}}^k - \boldsymbol{u}^k||_{L^2} \le ||\hat{\boldsymbol{u}}^k - \tilde{\boldsymbol{u}}^k||_{L^2} + ||\tilde{\boldsymbol{u}}^k - \boldsymbol{u}^k||_{L^2}$$

where $\tilde{\boldsymbol{u}}^k$ is the piecewise constant projection of $\boldsymbol{u}^k$ on mesh D_h and is given by

$$\tilde{\boldsymbol{u}}^k(\boldsymbol{x}) := \sum_{i, \boldsymbol{x}_i \in D} \tilde{\boldsymbol{u}}_i^k \chi_{U_i}(\boldsymbol{x})$$

where

$$\tilde{\boldsymbol{u}}_i^k := \frac{1}{h^d} \int_{U_i} \boldsymbol{u}^k(\boldsymbol{x}) d\boldsymbol{x}.$$

Finite difference approximation...

The error between projection of $\boldsymbol{u}^k$ and $\boldsymbol{u}^k$ is controlled by mesh size and satisfies

$$||\tilde{\boldsymbol{u}}^k - \boldsymbol{u}^k||_{L^2} = O(h^\gamma)$$

Consistency error: Denote $e_i^k(u) := \hat{\boldsymbol{u}}_i^k - \tilde{\boldsymbol{u}}_i^k$ and $e_i^k(v) := \hat{\boldsymbol{v}}_i^k - \tilde{\boldsymbol{v}}_i^k$. They satisfy following equation

$$\begin{aligned}
\boldsymbol{e}_i^{k+1}(u) &= \boldsymbol{e}_i^k(u) + \Delta t \boldsymbol{e}_i^k(v) + \Delta t \tau_i^k(u), \\
\boldsymbol{e}_i^{k+1}(v) &= \boldsymbol{e}_i^k(v) + \Delta t \left(\tau_i^k(v) + \sigma_i^k(u) + \sigma_i^k(v)\right) \\
&\quad + \Delta t \underbrace{\left(\boldsymbol{f}^\epsilon(\hat{\boldsymbol{u}}^k)(\boldsymbol{x}_i) + \boldsymbol{f}^\epsilon(\tilde{\boldsymbol{u}}^k)(\boldsymbol{x}_i)\right)}_{L^2 \text{ norm is bounded by } ||\hat{\boldsymbol{u}}^k - \tilde{\boldsymbol{u}}^k||_{L^2} = ||e^k(u)||_{L^2}}.
\end{aligned}$$

$$\begin{aligned}
\tau_i^k(v) &:= \frac{\partial \tilde{\boldsymbol{v}}_i^k}{\partial t} - \frac{\tilde{\boldsymbol{v}}_i^{k+1} - \tilde{\boldsymbol{v}}_i^k}{\Delta t} \\
\tau_i^k(v) &:= \frac{\partial \tilde{\boldsymbol{v}}_i^k}{\partial t} - \frac{\tilde{\boldsymbol{v}}_i^{k+1} - \tilde{\boldsymbol{v}}_i^k}{\Delta t} \\
\sigma_i^k(u) &:= \left(\boldsymbol{f}^\epsilon(\tilde{\boldsymbol{u}}^k)(\boldsymbol{x}_i) + \boldsymbol{f}^\epsilon(\boldsymbol{u}^k)(\boldsymbol{x}_i)\right) \\
\sigma_i^k(v) &:= \frac{\partial \boldsymbol{v}_i^k}{\partial t} - \frac{\partial \tilde{\boldsymbol{v}}_i^k}{\partial t}.
\end{aligned}
\longrightarrow
\begin{aligned}
||\tau^k(v)||_{L^2} &= O(\Delta t) \\
||\tau^k(u)||_{L^2} &= O(\Delta t) \\
||\sigma^k(u)||_{L^2} &= O(h^\gamma/\epsilon^2) \\
||\sigma^k(v)||_{L^2} &= O(h^\gamma)
\end{aligned}$$

Discussion

- We can extend the procedure to implicit scheme and derive similar result.
- Stability of perturbation shows that Forward Euler is stable only near $\Delta t \to 0$.
- Backward Euler is unconditionally stable only if the strain $S(\boldsymbol{y}, \boldsymbol{x}; \boldsymbol{u}) < S_c$. S_c is the critical strain at which the deformation enters strain softening region.
- For exact solution in $C^1(D; \mathbb{R}^d)$ the rate of convergence is $O(h/\epsilon^2)$. This restricts the choice of h to be very small.

Numerical analysis in one dimension

Let $D = (a, b)$ be material domain and $J = [0, T]$ be time domain. $D_\epsilon = [a - 2\epsilon, a] \cup [b, b + 2\epsilon]$.

We have following goals

- To see if we can improve the rate of convergence and find the condition on Δt for which scheme is stable.
- Compute the rate of convergence of nonlinear peridynamic solution to linear ealstic solution.

Let u^ϵ denote the solution of nonlinear peridynamics equation, u_l^ϵ denote the solution of linear peridynamic equation, and let u denote the solution of linear elastic equation.

$$\begin{aligned}
\rho \ddot{u}(t, x) &= \mathbb{C} u_{xx}(t, x) + b(t, x) \\
\rho \ddot{u}^\epsilon(t, x) &= f^\epsilon(u^\epsilon(t))(x) + b(t, x) \\
\rho \ddot{u}_l^\epsilon(t, x) &= f_l^\epsilon(u_l^\epsilon(t))(x) + b(t, x)
\end{aligned}$$

$u^\epsilon, u_l^\epsilon, u$ satisfy identical boundary condition $u = 0$ on D_ϵ and identical initial condition $u = u_0$ and $\dot{u} = v_0$.

Numerical analysis in one dimension...

Strain in 1-d is defined as

$$S(y,x;u) := \frac{u(y)-u(x)}{|y-x|}$$

Nonlinear and linear peridynamic forces are given by

$$f^\epsilon(u)(x) = \frac{2}{\epsilon^2}\int_{x-\epsilon}^{x+\epsilon} J(|y-x|/\epsilon)\psi'(|y-x|S^2)S(y,x;u)dy$$
$$f_l^\epsilon(u)(x) = \frac{2}{\epsilon^2}\int_{x-\epsilon}^{x+\epsilon} J(|y-x|/\epsilon)\psi'(0)S(y,x;u)dy$$

Linear elastic constant $\mathbb{C}$ is defined as

$$\begin{aligned}\mathbb{C} &= \int_{-1}^{1} J(|z|)f'(0)|z|dz \\ &= \frac{1}{\epsilon^2}\int_{x-\epsilon}^{x+\epsilon} J(|y-x|/\epsilon)f'(0)|y-x|dy \qquad \forall x, \epsilon > 0\end{aligned}$$

$\mathbb{C}$ is related to $f^\epsilon(u)(x)$ in the sense that $f^\epsilon(u)(x) \to \mathbb{C}u_{xx}(x)$.

Limit of peridynamics to elastodynamics

Proposition 4 *If* $u \in C^3(D;\mathbb{R})$ *then*

$$\sup_{x\in D} |f^\epsilon(u)(x) - f_l^\epsilon(u)(x)| = O(\epsilon)$$
$$\sup_{x\in D} |f_l^\epsilon(u)(x) - \mathbb{C}u_{xx}(x)| = O(\epsilon)$$

Theorem 5 *Let* $e^\epsilon := u^\epsilon - u$. *Suppose* $u^\epsilon(t) \in C^4(D)$, *for all* $\epsilon > 0$ *and* $t \in [0,T]$. *Suppose there exists* $C_1 > 0$, C_1 *independent of size of horizon* ϵ, *such that*

$$\sup_{\epsilon>0}\left[\sup_{(x,t)\in D\times J} |u^\epsilon_{xxxx}(t,x)|\right] < C_1 < \infty$$

Then, $\exists C_2 > 0$ *such that*

$$\sup_{t\in[0,T]}\left\{\int_D \rho|\dot{e}^\epsilon(t,x)|^2 dx + \int_D \mathbb{C}|e^\epsilon_x(t,x)|^2 dx\right\} \le C_2\epsilon^2$$

and $u^\epsilon \to u$ *in* $H^1_0(D)$ *uniformly in time* $t \in [0,T]$.

Limit of peridynamics to elastodynamics...

Proof of theorem 5: $e^\epsilon := u^\epsilon - u$ satisfies following equation

$$\begin{aligned}\rho\ddot{e}^\epsilon(t,x) &= \mathbb{C}e^\epsilon_{xx}(t,x) + (-\boldsymbol{\nabla} PD^\epsilon(u^\epsilon(t))(x) - \mathbb{C}u^\epsilon_{xx}(t,x)) \\ &= \mathbb{C}e^\epsilon_{xx}(t,x) + F(t,x)\end{aligned}$$

Boundary condition $e^\epsilon(t,x) = 0$ on D_ϵ and initial condition $e^\epsilon(0,x) = 0, \dot{e}^\epsilon(0,x) = 0$ on D.

With hypothesis on u, we can show that

$$\sup_x |F(t,x)| \leq C_3\epsilon$$

where C_3 is independent of (t,x).

We multiply equation of e^ϵ by $\dot{e}^\epsilon$ and integrate over D to get

$$\begin{aligned}&\int_D \rho\ddot{e}^\epsilon(t,x)\dot{e}^\epsilon(t,x)dx = \int_D \mathbb{C}e^\epsilon_{xx}(t,x)\dot{e}^\epsilon(t,x)dx + \int_D F(t,x)\dot{e}^\epsilon(t,x)dx \\ \Rightarrow &\int_D \rho\frac{1}{2}\frac{d}{dt}|\dot{e}^\epsilon(t,x)|^2dx = \int_D \mathbb{C}\frac{d}{dx}(e^\epsilon_x(t,x)\dot{e}^\epsilon(t,x))dx - \int_D \mathbb{C}e^\epsilon_x(t,x)\dot{e}^\epsilon_x(t,x)dx + \int_D F(t,x)\dot{e}^\epsilon(t,x)dx \\ \Rightarrow &\frac{d}{dt}\left[\int_D \rho\frac{1}{2}|\dot{e}^\epsilon(t,x)|^2dx + \int_D \mathbb{C}\frac{1}{2}(e^\epsilon_x(t,x)e^\epsilon_x(t,x))dx\right] = \int_D F(t,x)\dot{e}^\epsilon(t,x)dx\end{aligned}$$

Limit of peridynamics to elastodynamics...

Also

$$\begin{aligned}\int_D F(t,x)\dot{e}^\epsilon(t,x)dx &= \int_D (\frac{1}{\sqrt{\rho}}F(t,x))(\sqrt{\rho}\dot{e}^\epsilon(t,x))dx \\ &\leq \int_D \rho/2|\dot{e}^\epsilon(t,x)|^2 dx + \int_D \frac{1}{2\rho}|F(t,x)|^2 dx \\ &\leq \int_D \rho\frac{1}{2}|\dot{e}^\epsilon(t,x)|^2 dx + \int_D \mathbb{C}\frac{1}{2}(e^\epsilon_x(t,x)e^\epsilon_x(t,x))dx + \int_D \frac{1}{2\rho}|F(t,x)|^2 dx\end{aligned}$$

Let

$$\eta(t) := \int_D \rho\frac{1}{2}|\dot{e}^\epsilon(t,x)|^2 dx + \int_D \mathbb{C}\frac{1}{2}(e^\epsilon_x(t,x)e^\epsilon_x(t,x))dx$$

$$\xi(t) := \int_D \frac{1}{2\rho}|F(t,x)|^2 dx$$

Then

$$\dot{\eta}(t) \leq \eta(t) + \xi(t) \quad \Rightarrow \quad \eta(t) \leq \exp[-t]\left[\eta(0) + \int_0^t \xi(\tau)\exp[-\tau]d\tau\right]$$

Since $\xi(t) \geq 0$ and $t > 0$, we have

$$\int_0^t \xi(\tau)\exp[-\tau]d\tau \leq \int_0^t \xi(\tau)d\tau \leq \int_0^t \sup \xi(\tau)d\tau \leq TC_4\epsilon^2$$

Limit of peridynamics to elastodynamics...

Combining above and substituting the definition of η back, and taking L^∞ norm over time, we get

$$\sup_{t\in[0,T]}\left\{\int_D \rho|\dot{e}^\epsilon(t,x)|^2 dx + \int_D \mathbb{C}|e^\epsilon_x(t,x)|^2 dx\right\} \le C_2\epsilon^2$$

Now to show that $e^\epsilon \to 0$ in $H^1(D)$, we use above equation and Poincare inequality as follows

$$\begin{aligned}||e^\epsilon(t,x)||^2_{L^2(D)} &\le C_1||e^\epsilon_x(t,x)||^2_{L^2(D)}\\ &\le C_1C_2\sup_{t\in[0,T]}\left\{\int_D \rho|\dot{e}^\epsilon(t,x)|^2 dx + \int_D \mathbb{C}|e^\epsilon_x(t,x)|^2 dx\right\}\\ &\le C\epsilon^2\end{aligned}$$

Quadrature based finite element approximation

Let $D_h = D \cap h\mathbb{Z}$ be mesh and $D_{\epsilon,h} = D_\epsilon \cap h\mathbb{Z}$. Let $K \subset \mathbb{Z}$ be set of indices for which $ih \in D$ and K_ϵ for which $ih \in D_\epsilon$.

Let $\mathcal{I}_h[\cdot]$ be interpolation operator. For any function $g : D \cup D_\epsilon \to \mathbb{R}$ it is defined as

$$\mathcal{I}_h\left[g(y)\right] = \sum_{i \in K \cup K_\epsilon} g(x_i)\phi_i(y)$$

where ϕ_i is the interpolation function.

Piecewise constant interpolation

$$\phi_i(x) = \begin{cases} 1 & x \in [x_i - h/2, x_i + h/2), \\ 0 & \text{otherwise} \end{cases}$$

Linear interpolation

$$\phi_i(x) = \begin{cases} 0 & \text{if } x \notin [x_{i-1}, x_{i+1}], \\ \dfrac{x - x_{i-1}}{h} & \text{if } x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{h} & \text{if } x \in [x_i, x_{i+1}] \end{cases}$$

Approximation of force: Let f_h^ϵ denote approximation of f^ϵ and $f_{l,h}^\epsilon$ denote approximation of f_l^ϵ.

$$f_h^\epsilon(u)(x_i) = \frac{2}{\epsilon^2}\int_{x_i-\epsilon}^{x_i+\epsilon} \mathcal{I}_h\left[\psi'(|y - x_i|S(y, x_i; u)^2)S(y, x_i; u)\right] J(|y - x_i|/\epsilon)dy$$

$$f_{l,h}^\epsilon(u)(x_i) = \frac{2}{\epsilon^2}\int_{x_i-\epsilon}^{x_i+\epsilon} \mathcal{I}_h\left[\psi'(0)S(y, x_i; u)\right] J(|y - x_i|/\epsilon)dy$$

Quadrature based finite element approximation

We first consider spatially discretized equation. Later, we will discretize it in time and analyse stability.

Let $\{\hat{u}_i(t)\}_{i \in K \cup K_\epsilon}$ be the solution of following equation

$$\rho(x_i)\ddot{\hat{u}}_i(t) = f_h^\epsilon(\hat{u}_h(t))(x_i) + b(t,x_i) \qquad \forall i \in K \; \forall t \in [0,T]$$

where

$$\hat{u}_h(t,x) = \sum_{i \in K \cup K_\epsilon} \hat{u}_i(t)\phi_i(x) \qquad \forall x \in D$$

For linear peridynamic, we can write this in matrix form as follows

$$\ddot{U}_h(t) = AU_h(t) + B(t)$$

$A = (a_{ij})_{i,j \in K}$ with a_{ij} defined as

If $|i-j|$ such that $|x_i - x_j| \le \epsilon + h$ then

$$a_{|i-j|} = \frac{2}{\epsilon^2}\frac{1}{\rho_i}\frac{f'(0)}{|x_j - x_i|}\int_{x_i-\epsilon}^{x_i+\epsilon} \phi_j(y) J(|y-x_i|/\epsilon) dy$$

Otherwise

$$a_{|i-j|} = 0$$

$$A = \begin{bmatrix} -\sum_j a_j & a_1 & a_2 & a_3 & \dots & 0 \\ a_1 & -\sum_j a_j & a_1 & a_2 & \dots & 0 \\ a_2 & a_1 & -\sum_j a_j & a_1 & .. & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & \dots & a_1 & -\sum_j a_j & a_1 & a_2 \\ 0 & \dots & a_2 & a_1 & -\sum_j a_j & a_1 \\ 0 & \dots & a_3 & a_2 & a_1 & -\sum_j a_j \end{bmatrix}$$

Convergence and stability

Proposition 6 *Square matrix $-A$ of size $|K| \times |K|$ is Stieltjes matrix, i.e. it is an nonsingular symmetric M-matrix. Therefore, the eigenvalues of $-A$ of are real and strictly positive. Further, eigenvalues λ_i of $-A$ are bounded by*

$$\max_i \lambda_i \leq \frac{2}{\epsilon h} \frac{\mathbb{C}}{\rho}$$

where $\rho = \sup_x \rho(x) > 0$.

Time discretization: Central difference scheme

Let U_h^k denote the discrete displacement vector at time t^k. It satisfies following equation

$$U_h^{k+1} = -U_h^{k-1} + (2 + \Delta t^2) A U_h^k + \Delta t^2 B^k \qquad \forall i \in K$$

with $U_h^k = 0$ for $i \in K_\epsilon$ and U_h^0 satisfies the initial displacement condition. To start the iteration, i.e. at $k = 0$ we need displacement at $k = -1$. We use initial velocity to get over this.

Let $E^k := U_h^k - U^k$ be error vector. This satisfies following equation

$$E^{k+1} = -E^{k-1} + (2 + \Delta t^2 A) E^k + \Delta t^2 (\tau^k + \sigma^k)$$

$$\tau_i^k = \ddot{u}(t^k, x_i) - \frac{u(t^{k+1}, x_i) - 2u(t^k, x_i) + u(t^{k-1}, x_i)}{\Delta t^2}$$
$$\sigma_i^k = f_{l,h}^\epsilon(u(t^k))(x_i) + f_l^\epsilon(u(t^k))(x_i)$$

$$\tau_i^k = O(\Delta t^2)$$
$$\sigma_i^k = O(h^\alpha / \epsilon)$$

Convergence and stability...

h-convergence: Rate of convergence of linear peridynamic as $h \to 0$ keeping ϵ fixed

Theorem 7 *Let $\{-\lambda_i\}$ be $|K|$ number of eigen values of A where each $\lambda_i > 0$. Let $\lambda = \max\{\lambda_i\}$. Scheme is stable as long as Δt satisfies following bound*

$$\Delta t \leq \frac{2}{\sqrt{\lambda}}$$

If $u \in C^4([0,T]; C^3(D))$ then consistency error τ_i^k and σ_i^k satisifies

$$\tau_i^k = O(\Delta t^2)$$
$$\sigma_i^k = O(h^\alpha/\epsilon)$$

where $\alpha = 1$ for piecewise constant interpolation and $\alpha = 2$ for linear interpolation.

Convergence and stability...

m**-convergence: Rate of convergence of linear peridynamic to elastodynamics as** $h \leq \epsilon \to 0$

Let $U^k = (u(t^k, x_i))i \in K \cup K_\epsilon$ where u is the exact solution of elastodynamics equation. We now take error E^k as error between approximate solution of linear peridynamic and exact solution of elastodynamics, i.e.

$$E^k = U_h^k - U^k$$

u is the solution of elastodynamic equation given by

$$\ddot{u}(t,x) = \frac{1}{\rho}\left(\mathbb{C}u_{xx}(t,x) + b(t,x)\right)$$

Theorem 8 *If $u \in C^4([0,T]; C^3(D))$ then consistency error τ_i^k and σ_i^k satisifies*

$$\tau_i^k = \ddot{u}(t^k, x_i) - \frac{u(t^{k+1}, x_i) - 2u(t^k, x_i) + u(t^{k-1}, x_i)}{\Delta t^2} = O(\Delta t^2)$$

For constant interpolation, we have

$$\sigma_i^k = f_{l,h}^\epsilon(u(t^k))(x_i) - \mathbb{C}u_{xx}(t^k, x_i) = O(\epsilon) + O(h)$$

For linear interpolation, we have

$$\sigma_i^k = f_{l,h}^\epsilon(u(t^k))(x_i) - \mathbb{C}u_{xx}(t^k, x_i) = O(\epsilon) + O(h^2/\epsilon)$$

Numerical verification

h- convergence

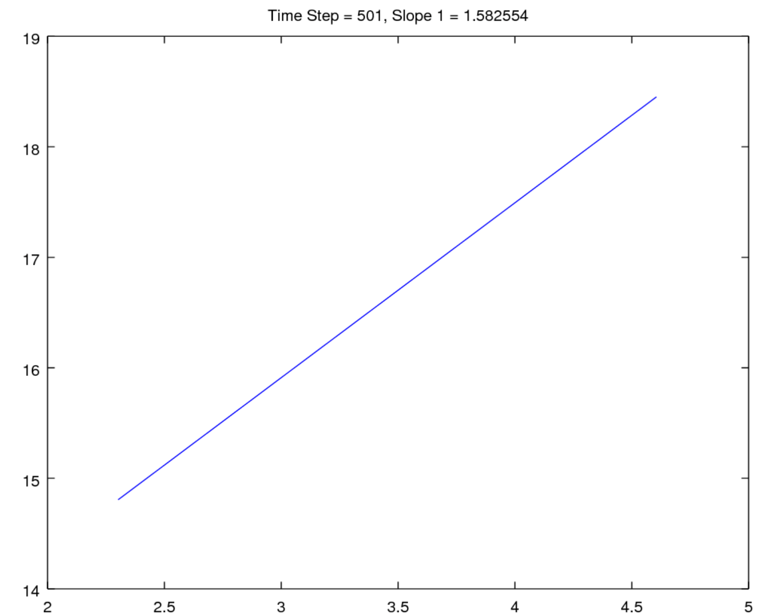
$\epsilon = 0.1$

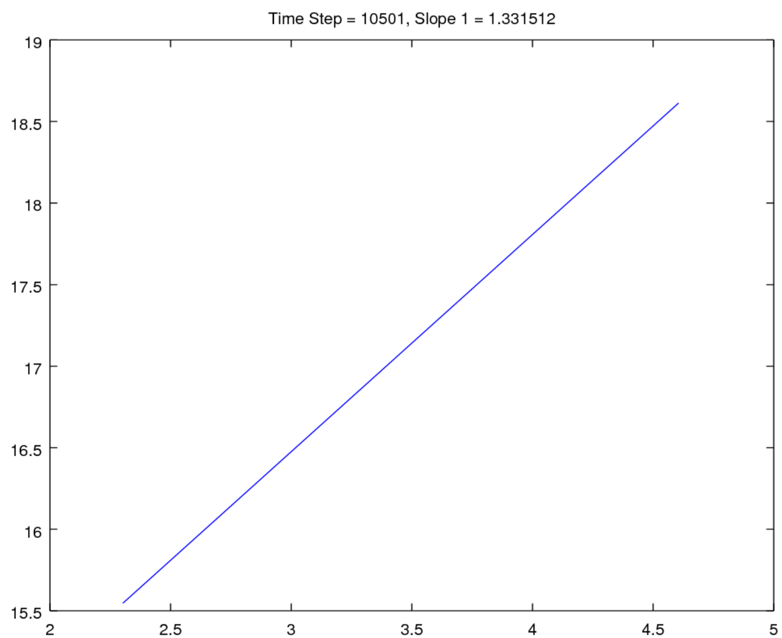
$h_i = \epsilon/10^i$, $i = 1, 2, 3$

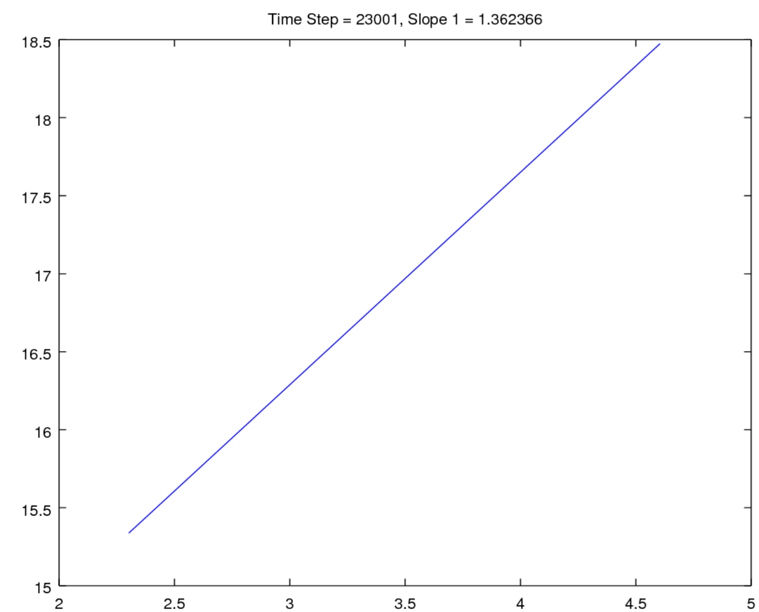
$\Delta t = 10^{-5}$

$\mathbb{C} = 20.0$

$\log ||u^i - u^{i-1}||$ vs $\log h_i/h_{i-1}$







We have used GNU-Octave and Visit, both open source, to generate the pictures and video.
Visit: https://wci.llnl.gov/simulation/computer-codes/visit
GNU-Octave: https://www.gnu.org/software/octave/

Numerical verification

Rate of convergence of difference of linear and nonlinear peridynamic solution

$\epsilon_i = 10^{-i-1}, h_i = \epsilon_i/10, i = 1, 2, 3$

$\Delta t = 10^{-6}$

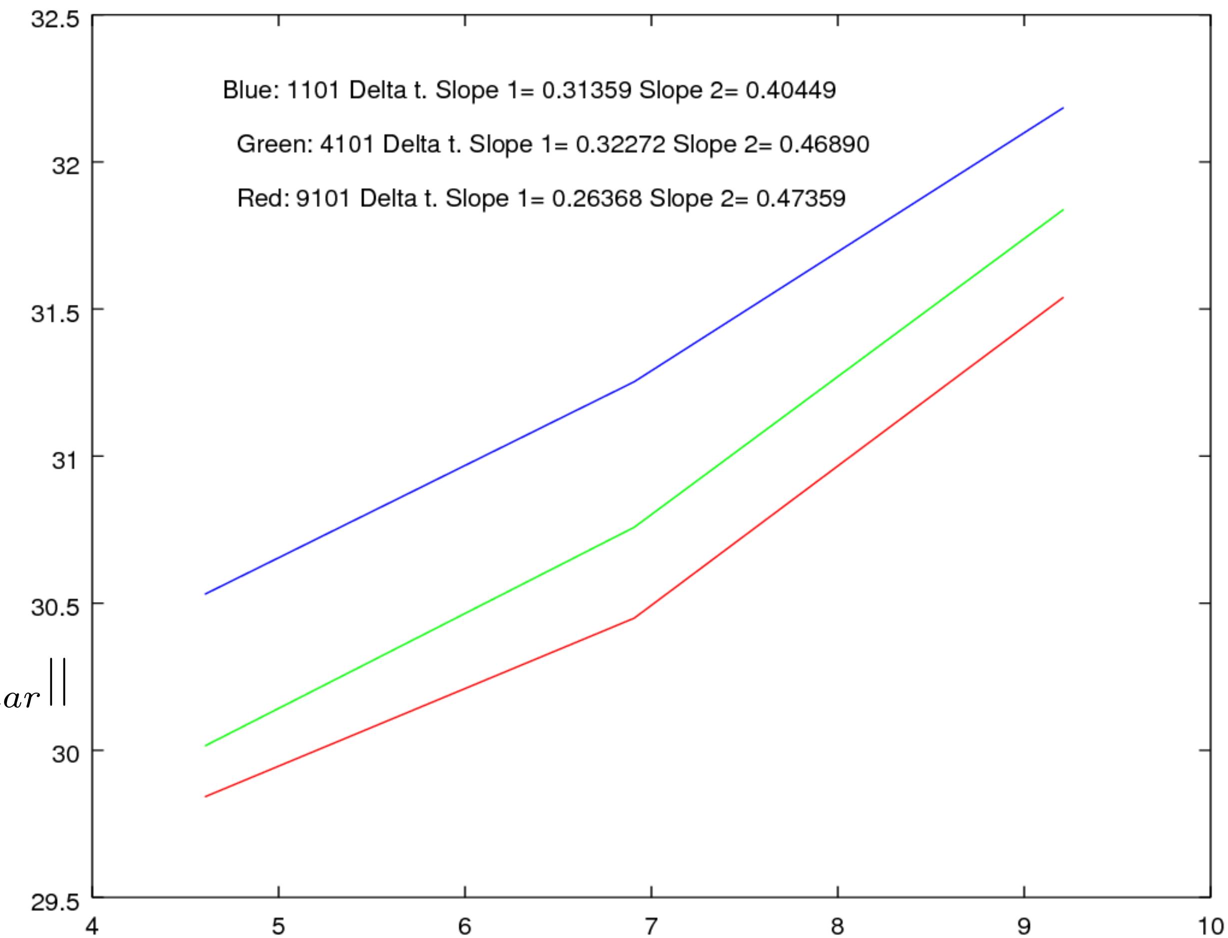
Linear interpolation

Gaussian IC with amplitude$= 10^{-5}$

$\psi'(r) = C(1 - \exp[-\beta r]), C = 2, \beta = 10$

$J(r) = 1$

$$Y = -\log ||u^i_{linear} - u^i_{nonlinear}||$$



$$X = -\log(\epsilon_i/10)$$

Wave dispersion

In case of linear ealsticity, we find that the for solution $u(t,x) = a\exp[i\kappa(x - vt)]$ the phase velocity comes out to be constant.

However, in peridynamic this is not the case and we have wave dispersion. Following simulation shows how wave dispersion takes place in the peridynamic.

$\epsilon = 0.00125, h = \epsilon/16, \Delta t = 10^{-6}$

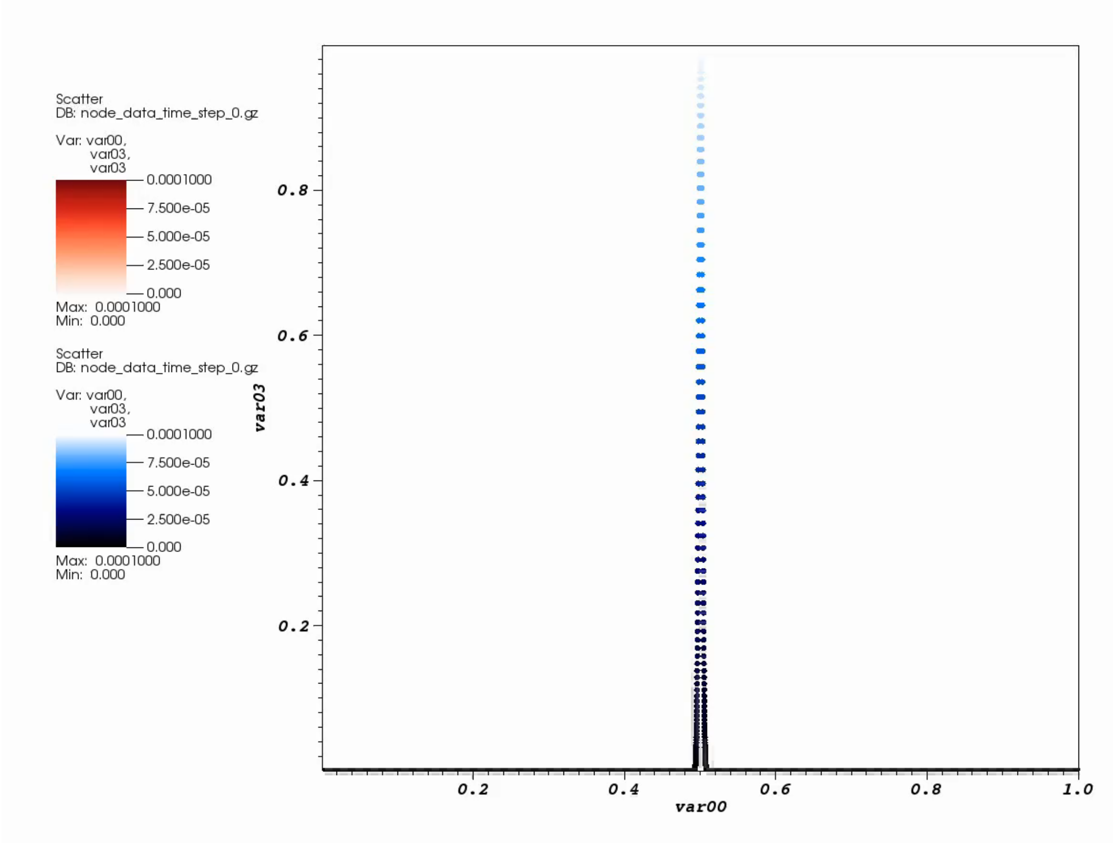
Linear interpolation

Gaussian IC with amplitude$= 10^{-4}$

$\psi'(r) = C(1 - \exp[-\beta r]), C = 2, \beta = 100$

$J(r) = 1$

Red: Linear peridynamics
Blue: Nonlinear Peridynamics



Wave dispersion...

$\epsilon = 0.00015625, h = \epsilon/4, \Delta t = 10^{-5}$

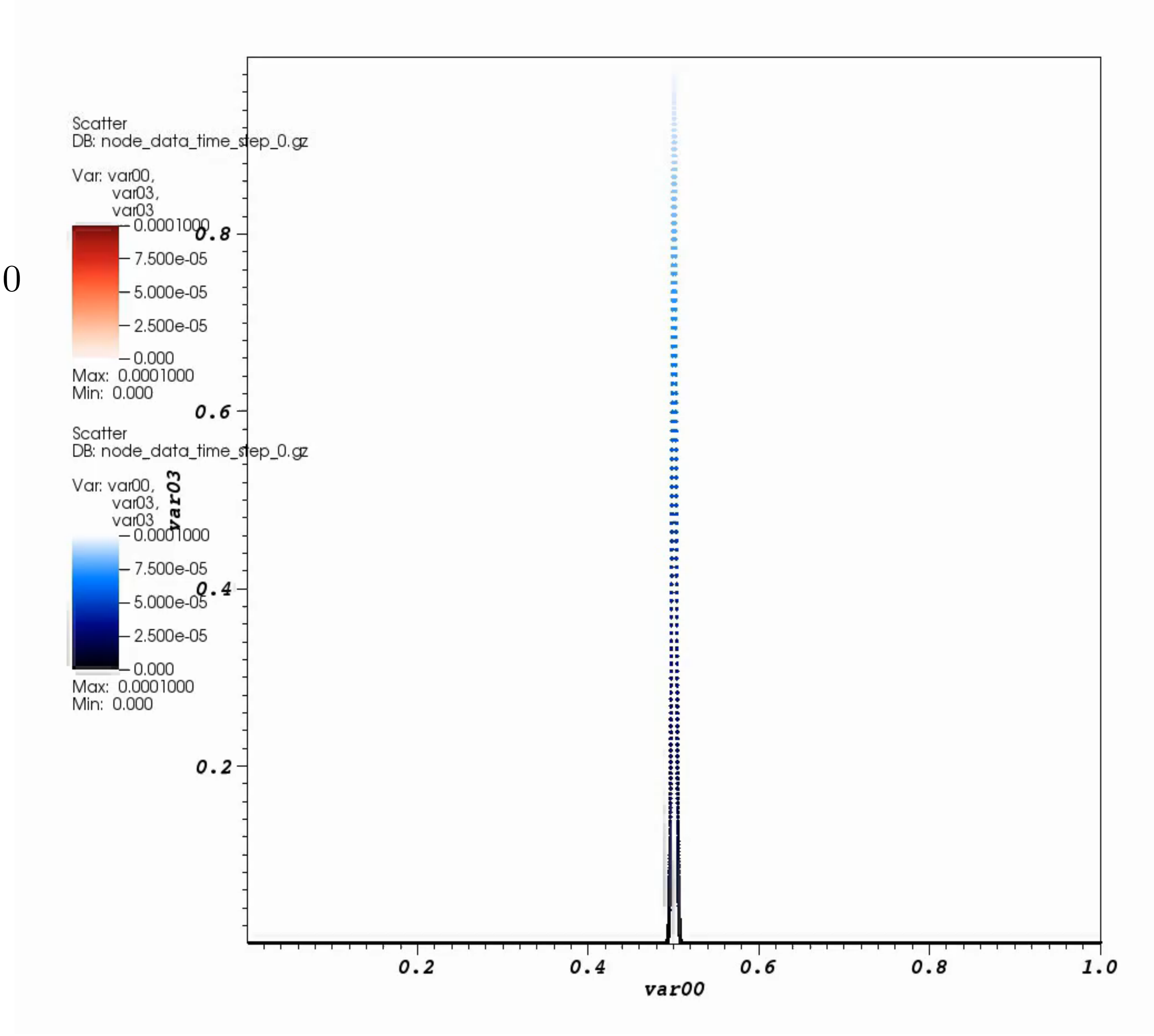
Linear interpolation

Gaussian IC with amplitude$= 10^{-4}$

$\psi'(r) = C(1 - \exp[-\beta r]), C = 2, \beta = 10$

$J(r) = 1$

Red: Linear peridynamics
Blue: Nonlinear Peridynamics



Wave dispersion.

$\epsilon = 0.001, h = \epsilon/10, \Delta t = 5 \times 10^{-6}$

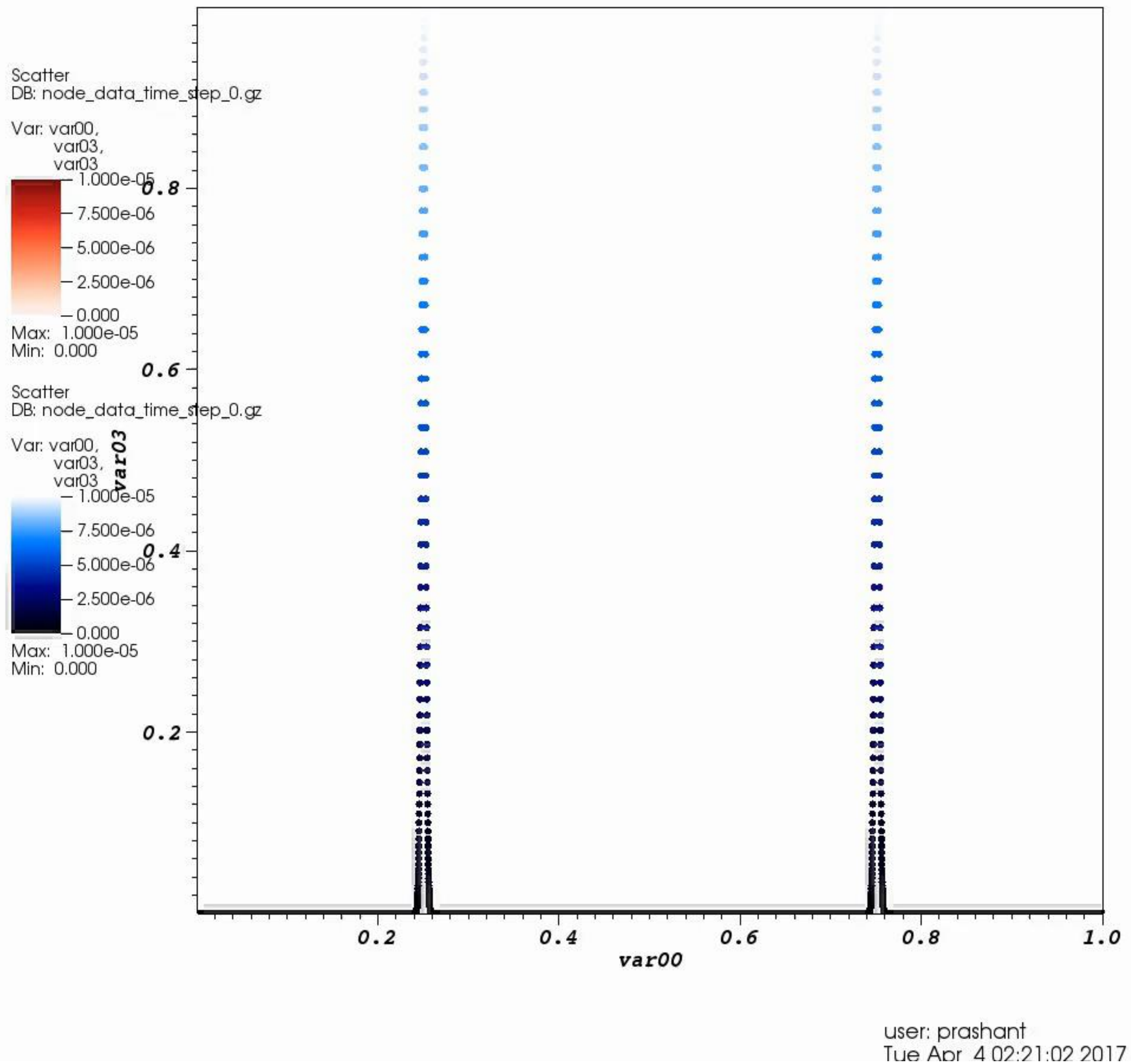
Linear interpolation

Double Gaussian IC with amplitude= 10^{-5}

$\psi'(r) = C(1 - \exp[-\beta r]), C = 2, \beta = 10$

$J(r) = 1$

Red: Linear peridynamics
Blue: Nonlinear Peridynamics



Wave dispersion...

Consider bar of infinite length. We substitute following displacement field in peridynamic equation

$$u(t,x) = a \exp[i\kappa(x - vt)]$$

and solve for phase velocity. It is given by

$$v^2 = -\frac{2}{\epsilon^2\kappa^2} \sum_{j=-\epsilon/h+1}^{\epsilon/h+1} a_{|j|}\psi' \left[|j|h \left(\frac{a\exp[i\kappa(jh - vt)] - a\exp[i\kappa(0 - vt)]}{|j|h} \right)^2 \right] \frac{\exp[i\kappa(jh)] - 1}{|j|h}$$

For nonlinear peridynamics, we need to solve above equation using nonlinear solver. However, for linear peridynamic, abve equation simplifies to

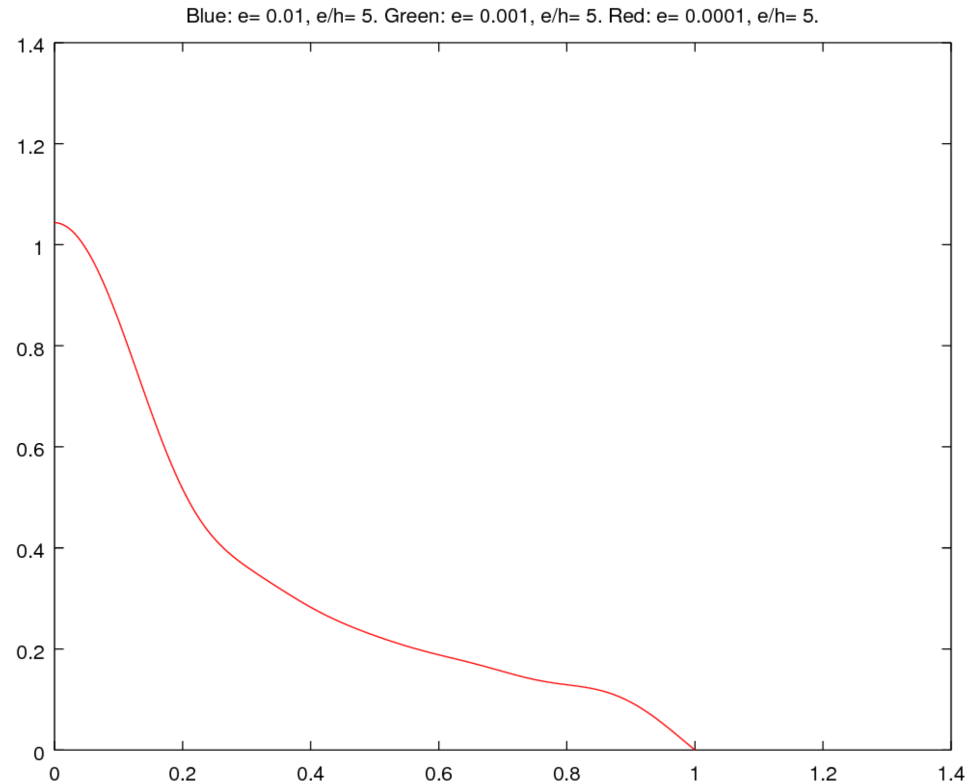
$$v^2 = -\frac{2\psi'(0)}{\epsilon^2\kappa^2} \sum_{j=-\epsilon/h+1}^{\epsilon/h+1} a_{|j|} \frac{\exp[i\kappa(jh)] - 1}{|j|h}$$

Wave dispersion: Phase velocity as a function of k

$\epsilon_i = 10^{-i-1}, h_i = \epsilon_i/5, i = 1, 2, 3$

$J(r) = 1$

$$Y = \frac{v}{\psi'(0)}$$



$$X = \frac{\kappa h}{2\pi}$$

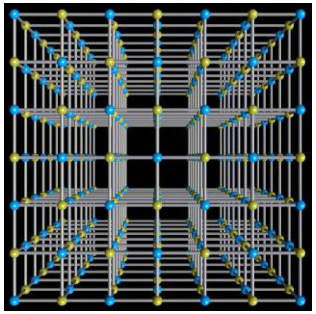
This result agrees with the **Bazant, Luo, Chau, and Bessa 2016** *Wave dispersion and basic concepts of peridynamics compared to classical nonlocal damage models.*

Discussion

- In 1-d, we find that we can select h in the same order as size of horizon and still get O(h) convergence in space.
- Central difference is shown to be stable for linear Peridynamics. If the deformation in material is such that the strain remains away from softening zone then for such deformation difference of linear and nonlinear Peridynamic force is of the order of size of horizon. Therefore, the stability of nonlinear model is also guaranteed.
- We also see from our simulation that difference between linear and nonlinear Peridynamic evolution is very small for small elastic constant and small horizon.

Future works

- Extension of 1-d method to higher dimension.
- Further investigation of numerical method is needed for the case when strain enters softening zone.
- Extension to state based Peridynamic models.



Thank you!