Numerical fracture experiments using nonlinear nonlocal models

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Joint work with
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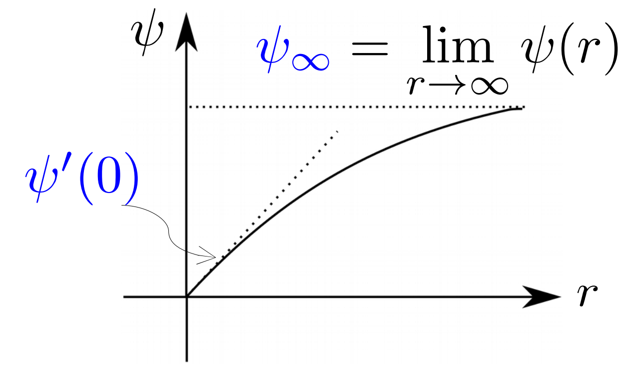
Overview of the talk

- Nonlinear peridynamic model
- Well-posedness of nonlocal model
- Numerical results
- Ongoing and future work

Nonlinear peridynamic model

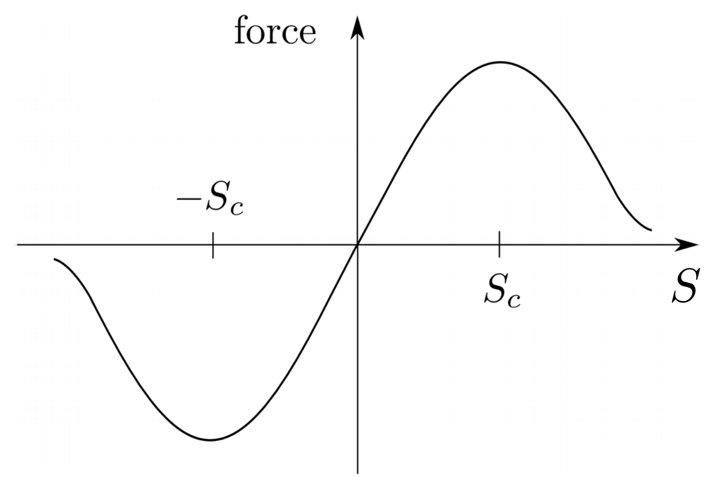
Let ϵ is the horizon, $B_\epsilon(\boldsymbol{x})$ ball of radius ϵ, and $\boldsymbol{u}(\boldsymbol{x})$ displacement of material point $\boldsymbol{x} \in D$. In this work, we consider linearized pairwise strain $S(\boldsymbol{y}, \boldsymbol{x}; \boldsymbol{u})$ given by

$$S(\boldsymbol{y}, \boldsymbol{x}; \boldsymbol{u}) = \frac{\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})}{|\boldsymbol{y} - \boldsymbol{x}|} \cdot \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|}.$$



Suppose $\hat{\boldsymbol{f}}^\epsilon(\boldsymbol{y}, \boldsymbol{x})$ denotes the force applied on $\boldsymbol{x}$ from the neighboring point $\boldsymbol{y}$. Then total force at $\boldsymbol{x}$ is given by

$$\boldsymbol{f}^\epsilon(\boldsymbol{x}) = \int_{B_\epsilon(\boldsymbol{x})} \hat{\boldsymbol{f}}^\epsilon(\boldsymbol{y}, \boldsymbol{x}) d\boldsymbol{y}$$



We consider pairwise force based on smooth and concave potential function ψ[1,2]

$$\hat{\boldsymbol{f}}^\epsilon(\boldsymbol{y}, \boldsymbol{x}) = \frac{1}{\epsilon |B_\epsilon(\boldsymbol{0})|} \frac{\partial_S \psi(|\boldsymbol{y} - \boldsymbol{x}| S(\boldsymbol{y}, \boldsymbol{x})^2)}{|\boldsymbol{y} - \boldsymbol{x}|} \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|}$$

[1] R. Lipton (2014) Dynamic brittle fracture as a small horizon limit of peridynamics. Journal of Elasticity, 117(1) 21-50.
[2] R. Lipton (2016) Cohesive dynamics and brittle fracture. Journal of Elasticity, 124(2), pp.143-191.

Equation of motion

Equation of motion

$$\rho \ddot{\boldsymbol{u}}(\boldsymbol{x},t) = \boldsymbol{f}^{\epsilon}(\boldsymbol{x};\boldsymbol{u}(t)) + \boldsymbol{b}(\boldsymbol{x},t), \qquad \forall \boldsymbol{x} \in D, t \in [0,T]$$

Boundary condition

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{g}(\boldsymbol{x},t) \qquad \forall \boldsymbol{x} \in D_u, t \in [0,T]$$

$$\boldsymbol{b}(\boldsymbol{x},t) = \boldsymbol{f}_{ext}(\boldsymbol{x},t) \qquad \forall \boldsymbol{x} \in D_f, t \in [0,T]$$

$D_u, D_f \subset D$ are layer with finite volume (area in 2-d) on which displacement and external force, respectively, are specified. External force is applied in the form of body force.

Initial condition: $\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \dot{\boldsymbol{u}}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x})$ for all $\boldsymbol{x} \in D$.

Well-posedness of nonlinear peridynamic model

- Using the fact that nonlinear peridynamic force is bounded and Lipschitz continuous with respect to displacement field $\boldsymbol{u} \in L^2_0(D)$, the existence of solutions over any finite time domain $[0, T]$ is shown [1].

- To prove existence of solutions in more regular spaces, we introduce boundary function ω into Peridynamic force. $\omega(\boldsymbol{x}) = 1$ in the interior and smoothly decays to 0 as $\boldsymbol{x}$ approaches boundary ∂D.

- To perform apriori error analysis of finite difference approximation, we consider Hölder space $C^{0,\gamma}_0(D)$, $\gamma \in (0, 1]$. In [2] we show existence of solutions in Hölder space $C^{0,\gamma}_0(D)$. In [3] we extend the results to state-based peridynamic models.

- For apriori error analysis of finite element approximation using continuous piecewise linear elements, we consider natural space $H^2(D) \cap H^1_0(D)$. In [4] we show existence of solutions in $H^2(D) \cap H^1_0(D)$.

[1] R. Lipton (2016) Cohesive dynamics and brittle fracture. Journal of Elasticity, 124(2), pp.143-191.

[2] P.K. Jha and R. Lipton (2018) Numerical analysis of nonlocal fracture models in Holder space. SIAM Journal on Numerical Analysis, 56(2), pp.906-941.

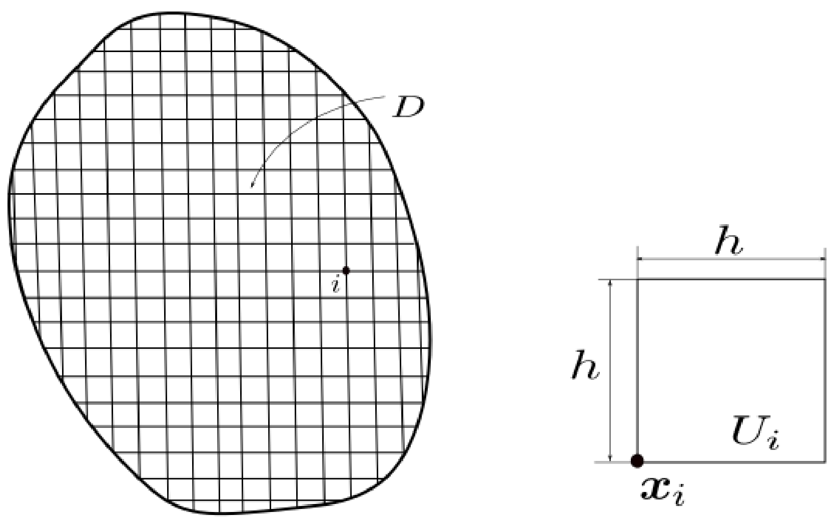
[3] P.K. Jha and R. Lipton (2019) Numerical convergence of finite difference approximations for state based peridynamic fracture models. Computer Methods in Applied Mechanics and Engineering, 351(1), 184 – 225.

[4] P.K. Jha and R. Lipton (2018) Finite element approximation of nonlocal fracture models. arXiv preprint arXiv:1710.07661. **Under review** in Discrete and Continuous Dynamical Systems Series B.

Finite difference approximation

We approximate peridynamic equation using piecewise constant interpolation and central in time discretization. Let $\boldsymbol{u}_i^k$ denote the discrete displacement at mesh node $\boldsymbol{x}_i$ and time $t^k = k\Delta t$. We consider following piecewise constant function

$$\boldsymbol{u}_h^k(\boldsymbol{x}) = \sum_{i,\boldsymbol{x}_i \in D} \boldsymbol{u}_i^k \chi_{U_i}(\boldsymbol{x})$$



(a) (b)

Discrete problem is

$$\frac{\boldsymbol{u}_h^{k+1} - 2\boldsymbol{u}_h^k + \boldsymbol{u}_h^{k-1}}{\Delta t^2} = \boldsymbol{f}_h^\epsilon(t^k) + \boldsymbol{b}_h^k,$$

where

$$\boldsymbol{f}_h^\epsilon(\boldsymbol{x}, t^k) = \sum_{i,\boldsymbol{x}_i \in D} \boldsymbol{f}^\epsilon(\boldsymbol{x}_i, t^k)\chi_{U_i}(\boldsymbol{x}),$$

$$\boldsymbol{b}_h(\boldsymbol{x}, t^k) = \sum_{i,\boldsymbol{x}_i \in D} \boldsymbol{b}(\boldsymbol{x}_i, t^k)\chi_{U_i}(\boldsymbol{x})$$

Convergence of finite difference approximation

Error at time step k is defined as: $E^k = ||\boldsymbol{u}_h^k - \boldsymbol{u}(t^k)||$.

Theorem 1. *Let $\epsilon > 0$ be fixed. Let $(\boldsymbol{u}, \boldsymbol{v})$ be the solution of peridynamic equation. We assume $\boldsymbol{u}, \boldsymbol{v} \in C^2([0,T]; C^{0,\gamma}(D;\mathbb{R}^d))$. Then the finite difference scheme is consistent in both time and spatial discretization and converges to the exact solution uniformly in time with respect to the L^2 norm. If we assume the error at the initial step is zero then the error E^k at time t^k is bounded and satisfies*

$$\sup_{0\leq k\leq T/\Delta t} E^k \leq O\left(C_t \Delta t + C_s \frac{h^\gamma}{\epsilon^2}\right),$$

where constant C_s and C_t are independent of h and Δt. Constants C_t, C_s depend on the ϵ and Hölder norm of the exact solution.

[1] P.K. Jha and R. Lipton (2018) Numerical analysis of nonlocal fracture models in Holder space. SIAM Journal on Numerical Analysis, 56(2), pp.906-941.

[2] P.K. Jha and R. Lipton (2019) Numerical convergence of finite difference approximations for state based peridynamic fracture models. Computer Methods in Applied Mechanics and Engineering, 351(1), 184 – 225.

Setting up peridynamic model

- Pairwise potential: $\psi(r) = c(1 - \exp[-\beta r^2])$
- Influence function: $J(r) = 1 - r$ for $0 \leq r < 1$ and $J(r) = 0$ for $r \geq 1$
- Critical strain: $S_c(\boldsymbol{y}, \boldsymbol{x}) = \dfrac{\pm \bar{r}}{\sqrt{|\boldsymbol{y} - \boldsymbol{x}|}}$, where $\bar{r}$ is the inflection point of function ψ
- We fix $\rho = 1200$ kg/m^3, bulk modulus $K = 3.24$ GPa, critical energy release rate $G_c = 500$ J/m^{-2}
- Using relation between nonlinear peridynamic model and linear elastic fracture mechanics[1], we find

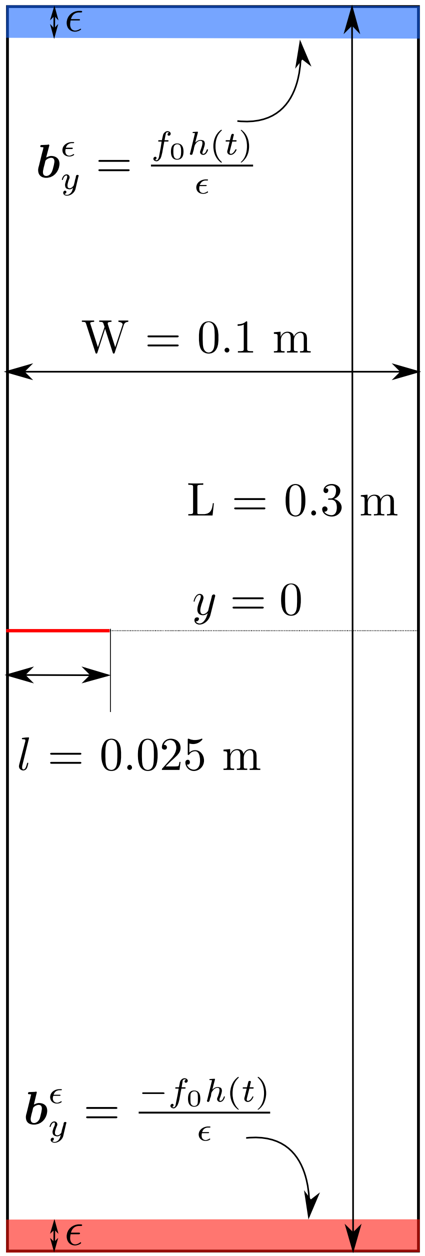
$$c = 4712.4, \qquad \beta = 1.7533 \times 10^8, \qquad \bar{r} = \frac{1}{\sqrt{2\beta}} = 5.3402 \times 10^{-5}$$

[1] R. Lipton (2016) Cohesive dynamics and brittle fracture. Journal of Elasticity, 124(2), pp.143-191.

Mode I crack propagation: Setup

Goal: Localization of crack and convergence to classical fracture mechanics for simple mode-I crack propagation[1]

- Final time $T = 560\,\mu$s, time step $\Delta t = 0.02\,\mu$s
- Uniform grid on square domain $D = [0, 0.1\,\text{m}] \times [-0.15\,\text{m}, 0.15\,\text{m}]$
- Experiment with three different horizons $\epsilon = 2.5, 1.25, 0.625\,\text{mm}$
- Body force $\boldsymbol{b}^\epsilon(\boldsymbol{x}, t) = (0, f_0 h(t)/\epsilon)$ on top layer and $\boldsymbol{b}^\epsilon(\boldsymbol{x}, t) = (0, -f_0 h(t)/\epsilon)$ on bottom layer
- $h(t)$ is a step function such that $h(t) = t$ for $t \leq 350\,\mu$s and $h(t) = 1$ for $t > 350\,\mu$s
- Mesh size is fixed by relation $h = \epsilon/4$

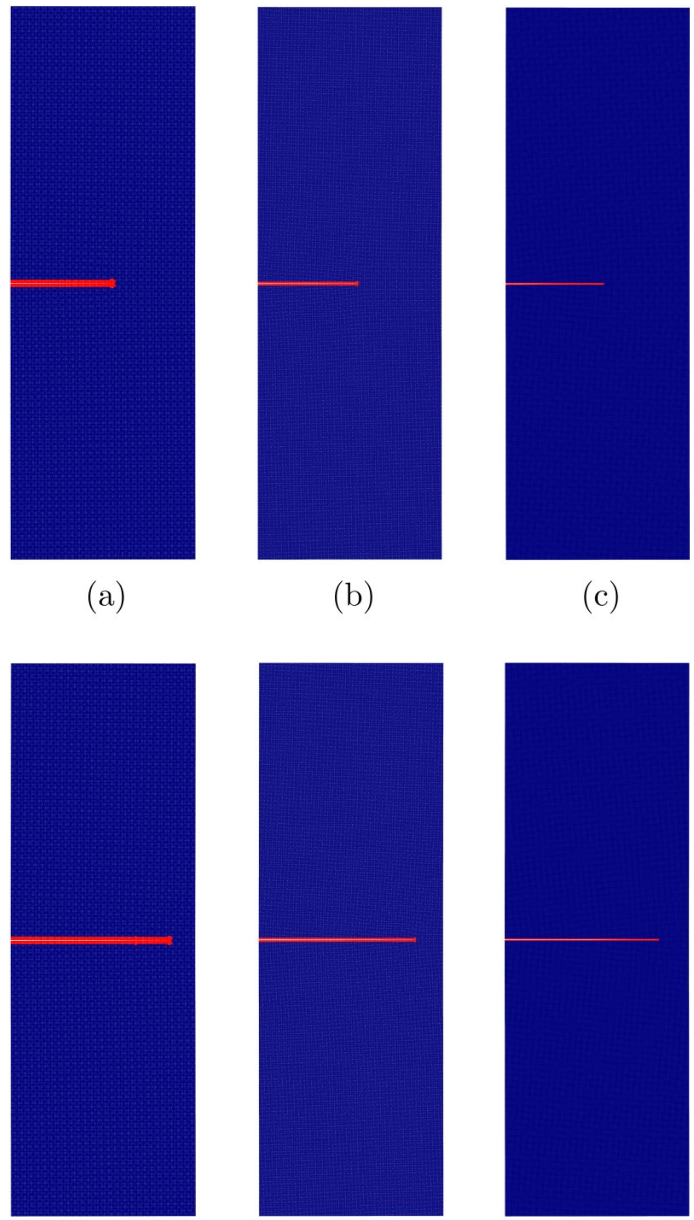


[1] R. Lipton & P.K. Jha (2019). The relation of nonlocal cohesive models to classic dynamic fracture models: The single edge notch in tension. ***In preparation***.

Localization of fracture zone

$t = 460\,\mu$s

$t = 520\,\mu$s

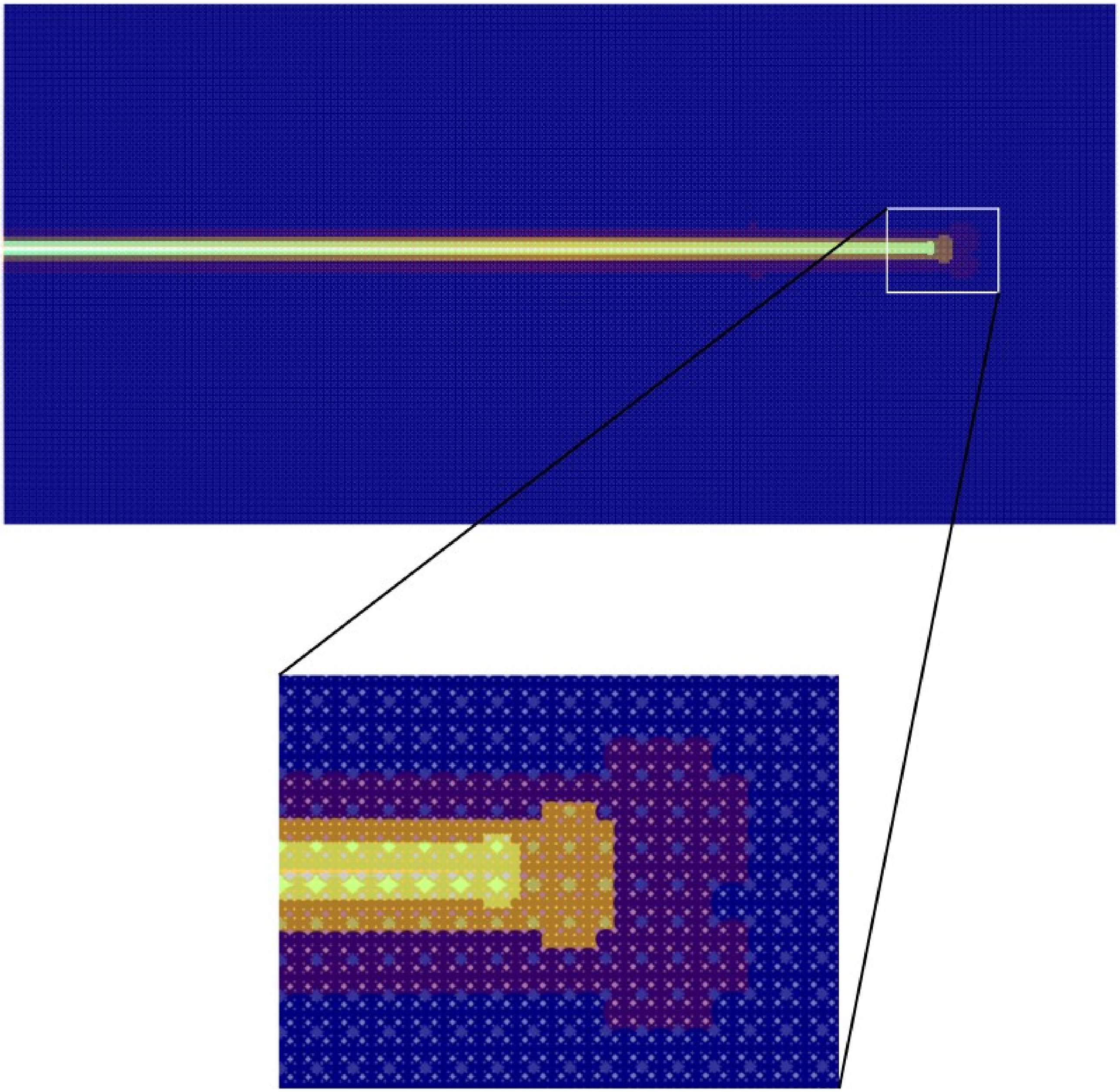


(a) (b) (c)

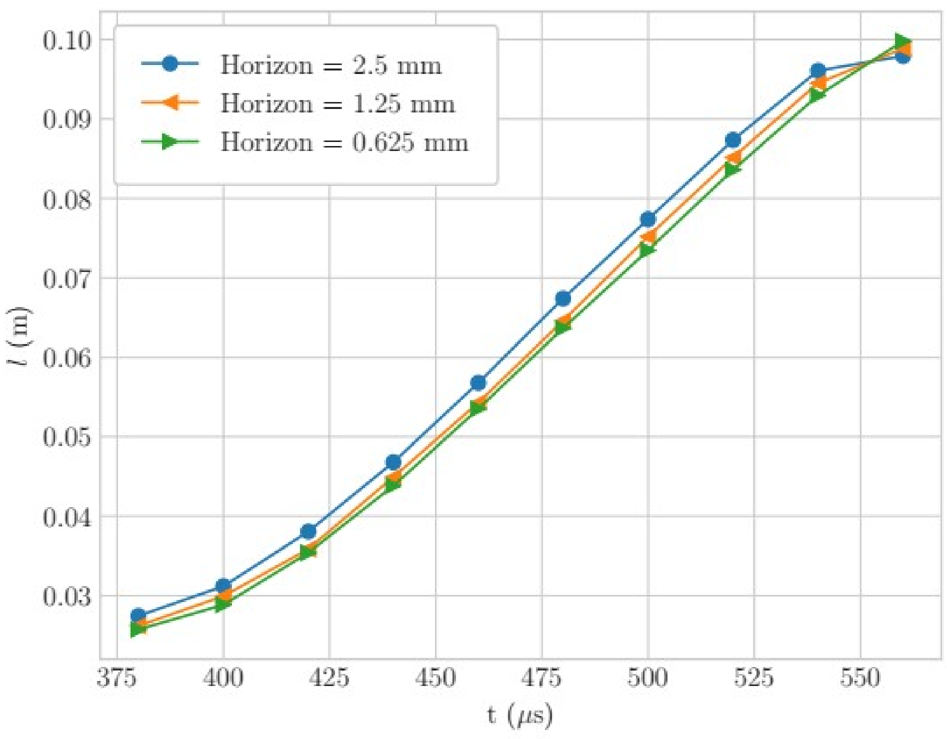
(d) (e) (f)

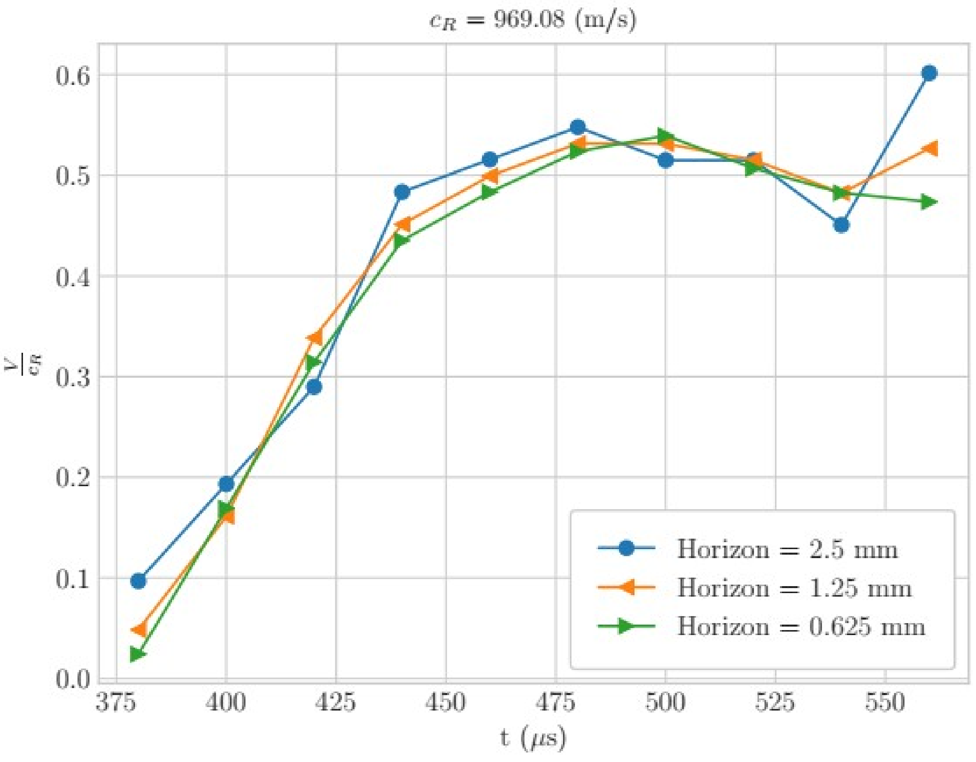
Localization of fracture zone

$t = 520\,\mu\text{s}$



Crack tip location and velocity



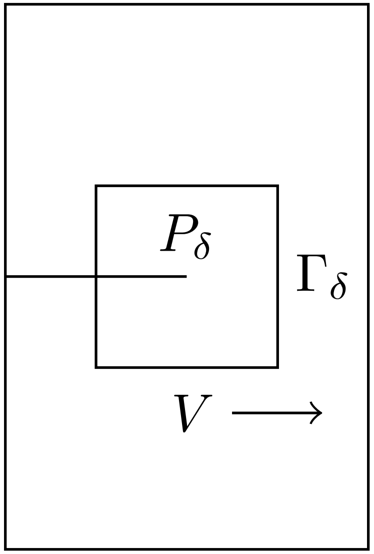


Energy into crack

The energy associated to crack is given by[1]

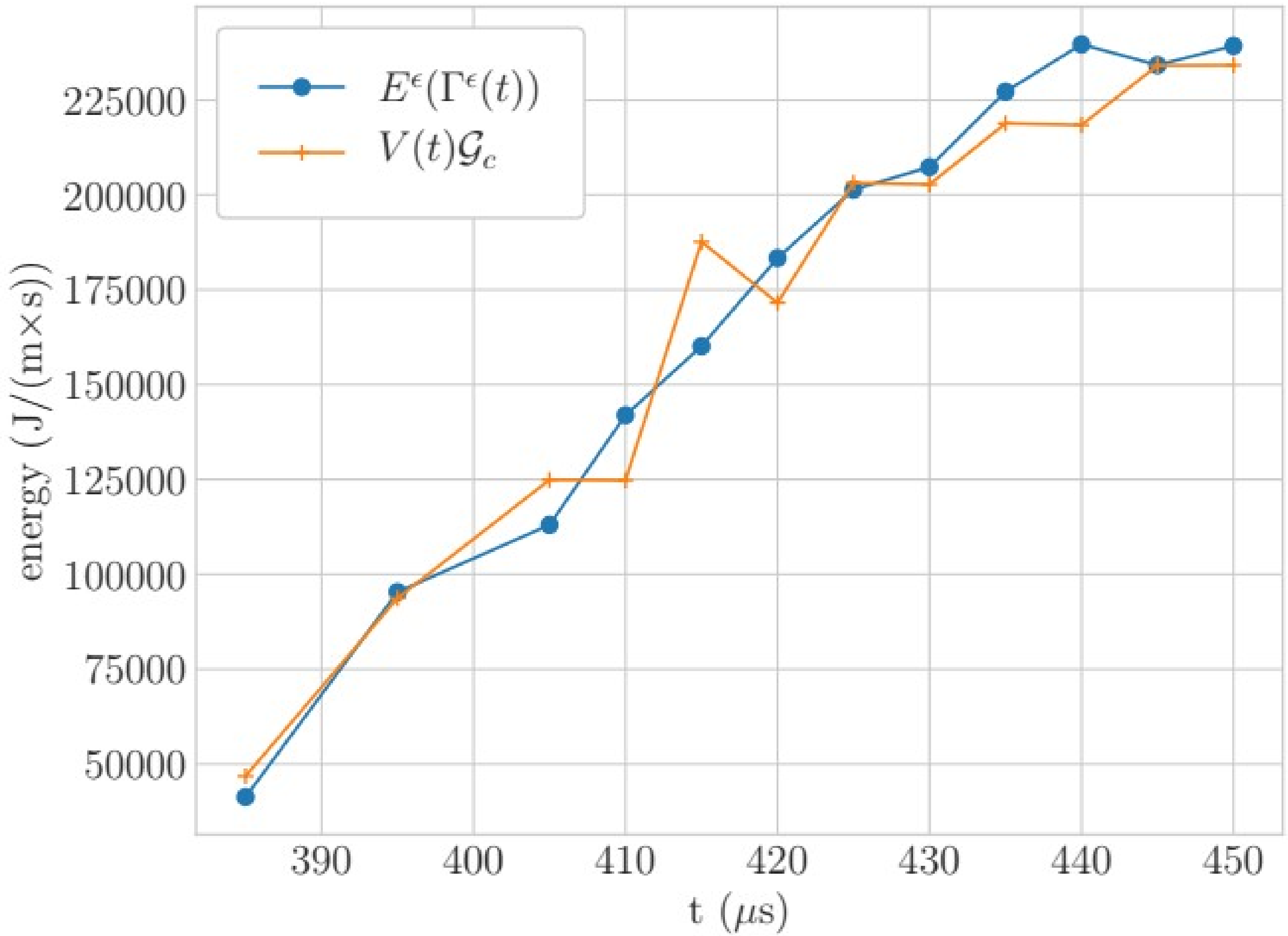
$$E(\Gamma_\delta(t)) = \frac{1}{|B_\epsilon(\mathbf{0})|}\int_{P_\delta^c(t)}\int_{P_\delta(t)\cap B_\epsilon(\boldsymbol{x})} \partial_S W(S(\boldsymbol{y},\boldsymbol{x};\boldsymbol{u}(t)))\frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|}\cdot(\dot{\boldsymbol{u}}(\boldsymbol{x},t)+\dot{\boldsymbol{u}}(\boldsymbol{y},t))d\boldsymbol{y}d\boldsymbol{x}.$$

Here W is the peridynamic pairwise energy density. $P_\delta(t)$ is the rectangle domain with crack tip at its center. It is moving with tip. $P_\delta^c(t)$ is the complement of $P_\delta(t)$.



[1] R. Lipton & P.K. Jha (2019). The relation of nonlocal cohesive models to classic dynamic fracture models: The single edge notch in tension. ***In preparation***.

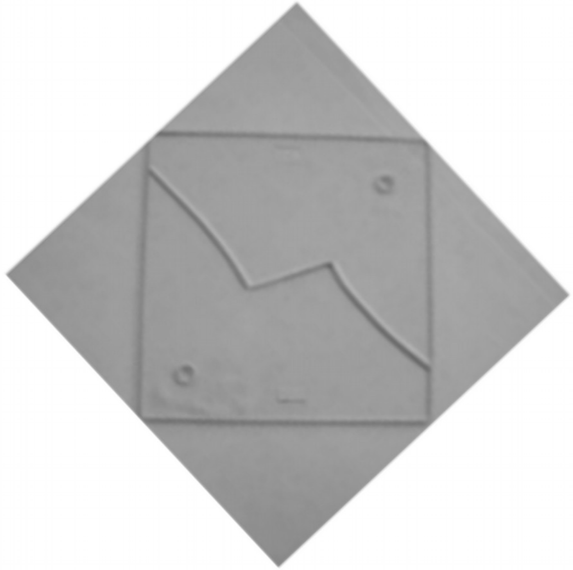
Energy into crack



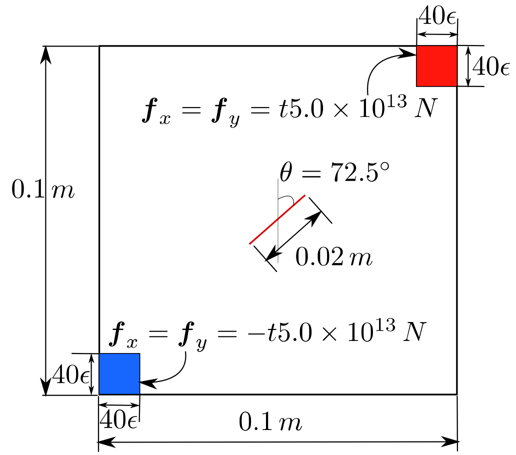
Mix mode crack propagation

Material properties are same as in the Mode-I problem. We set

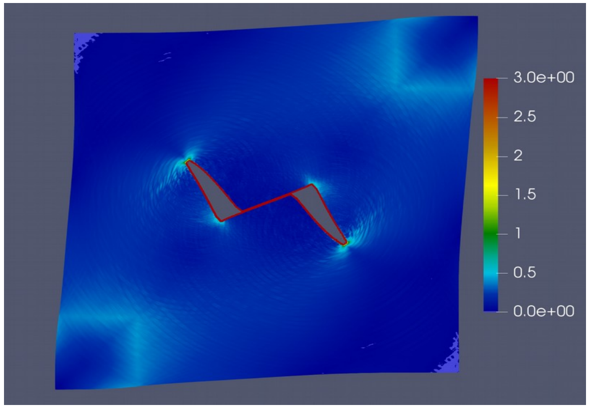
- Horizon $\epsilon = 0.5$ mm
- Mesh size $h = 0.125$ mm
- Final time $T = 140\,\mu$s
- Time step size $\Delta t = 0.004\,\mu$s



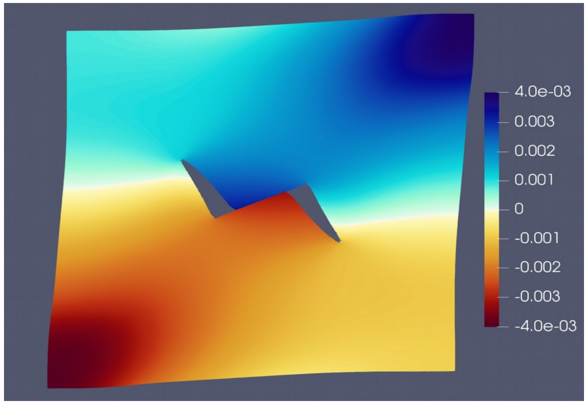
(e) Experiment result [2]



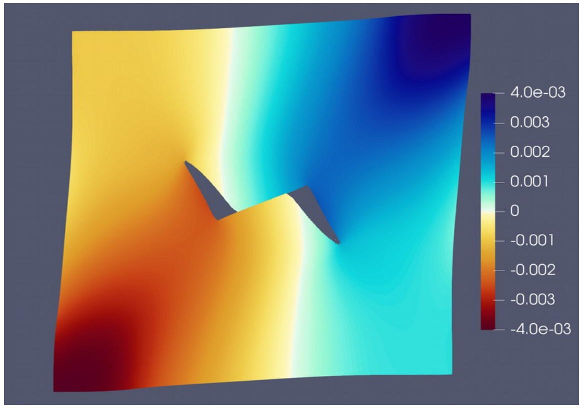
(a) Setup



(b) Damage profile



(c) $\boldsymbol{u}_x$ plot



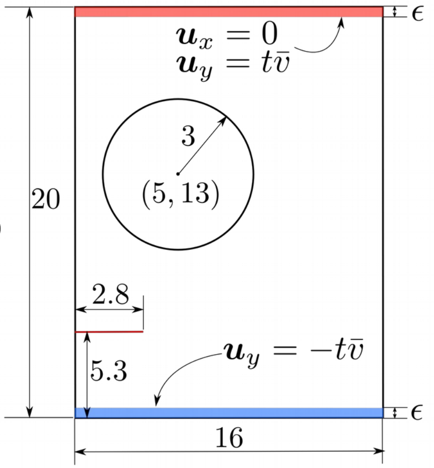
(d) $\boldsymbol{u}_y$ plot

[1] R. Lipton, R. Lehoucq, & P.K. Jha (2019) Complex fracture nucleation and evolution with nonlocal elastodynamics. Journal of Peridynamics and Nonlocal Modeling. April 2019.
[2] M. R. Ayatollahi & M. R. M. Aliha (2009). Analysis of a new specimen for mixed mode fracture tests on brittle materials. Engineering Fracture Mechanics, 76(11), 1563-1573.
[3] E. Madenci et al (2018). A state-based peridynamic analysis in a finite element framework. Engineering Fracture Mechanics, 195, pp.104-128.

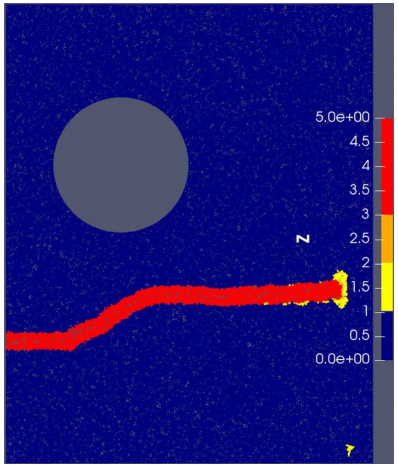
Crack-void interaction

Material properties are same as in the Mode-I problem. We set

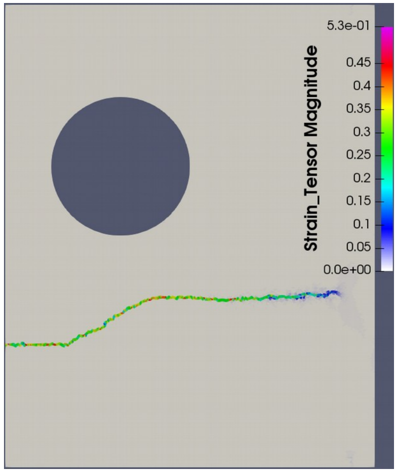
- Horizon $\epsilon = 0.4$ mm
- Mesh size $h = 0.1$ mm
- Final time $T = 800\,\mu$s
- Time step size $\Delta t = 0.004\,\mu$s



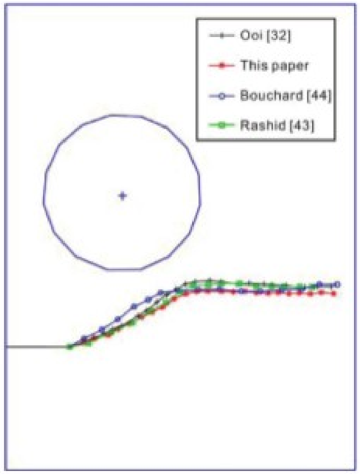
(a) Setup (units in mm)



(b) Damage profile



(c) Magnitude of symmetric gradient of displacement



(d) Numerical experiment results using FEM, Boundary element method [2]

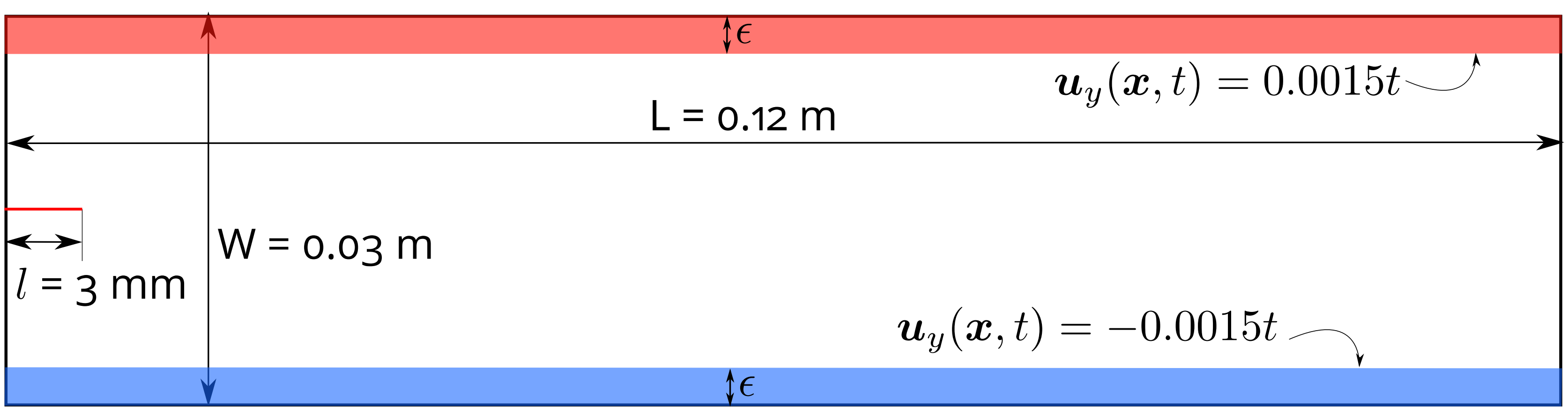
[1] P.K. Jha, P. Diehl & R. Lipton. Nodal finite element approximation of nonlocal fracture models. ***In preparation***.

[2] S. Dai, C. Augarde, C. Du & D. Chen (2015). A fully automatic polygon scaled boundary finite element method for modelling crack propagation. Engineering Fracture Mechanics, 133, 163-178.

Wave reflection effect on crack velocity

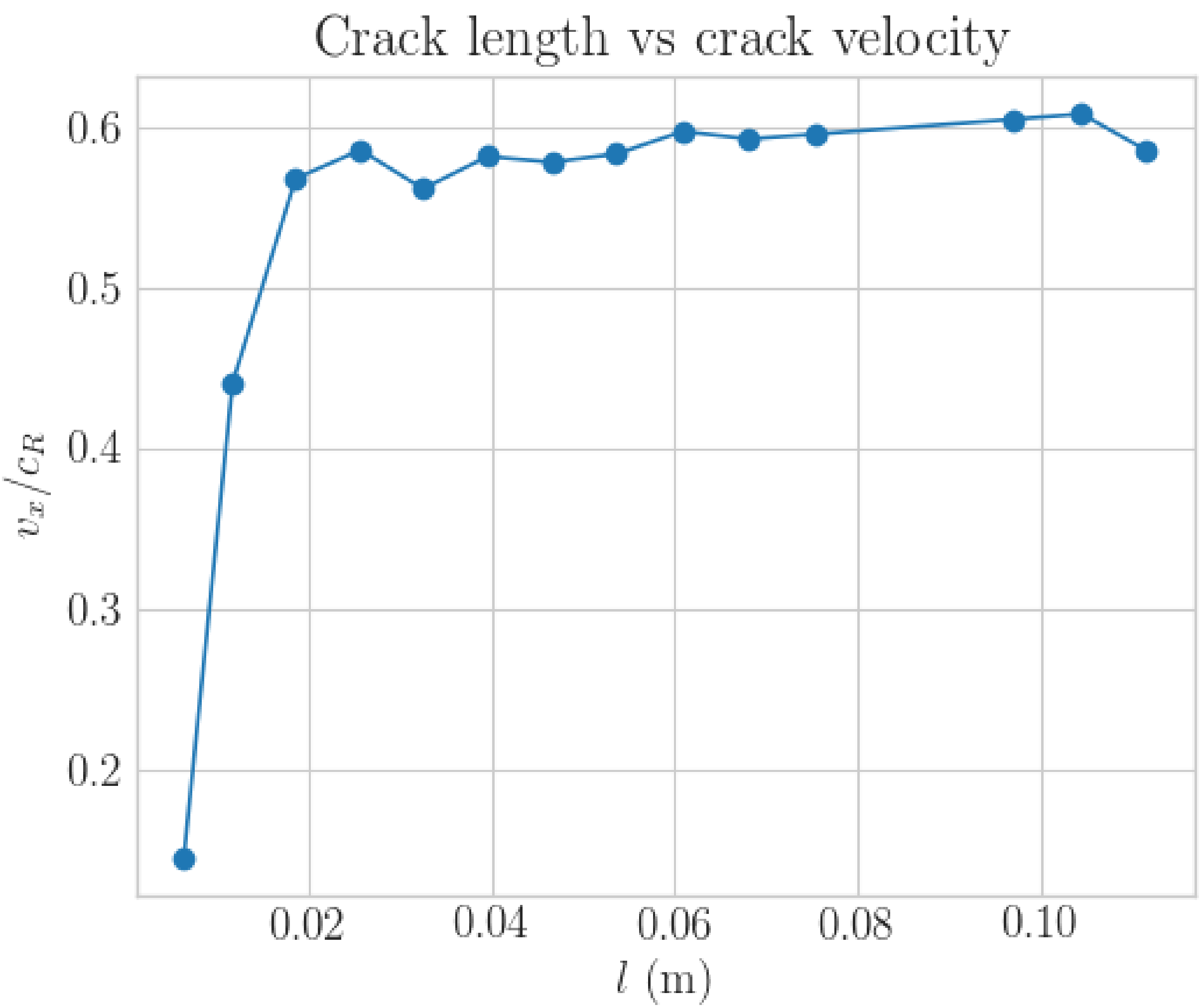
We consider a softer material with shear modulus $G = 35.2$ kPa, density $\rho = 1011$ kg/m^3, and critical energy release rate $G_c = 20$ J/m^{-2}. Poisson ratio is fixed to $\mu = 0.25$. Domain is $D = [0, 0.12\,\text{m}] \times [0, 0.03\,\text{m}]$.

- Horizon $\epsilon = 0.6$ mm, mesh size $h = 0.15$ mm
- Time $T = 1.1$ s, $\Delta t = 2.2\,\mu$s



Wave reflection effect on crack velocity

- Max crack length = 0.12 m
- Rayleigh wave speed $c_R = 5.502$ m/s



Ongoing and future works

- In [1] we show that the classical kinetic relation is embedded in peridynamics and we have $\lim_{\epsilon \to 0} J(t) = G_c$, where $J(t)$ is the nonlocal J-integral. In LEFM, the classical kinetic relation for the crack velocity is postulated. In contrast, we obtain the classical kinetic relation from the Peridynamics in the limit of vanishing nonlocality.

- Open source computational library for nonlocal modeling. This is a joint work with Patrick Diehl (LSU) and Robert Lipton (LSU).

- Study of granular material using nonlinear nonlocal model.

[1] R. Lipton & P.K. Jha (2019). The relation of nonlocal cohesive models to classic dynamic fracture models: The single edge notch in tension. ***In preparation***.

Thank you!