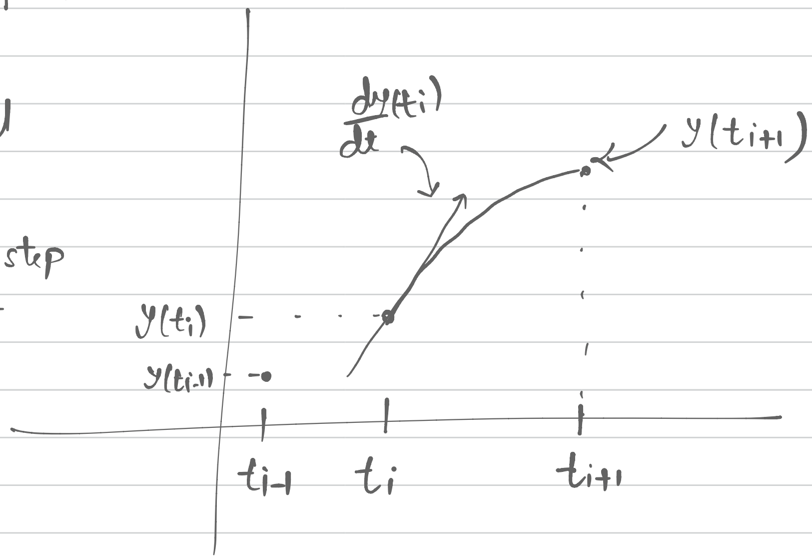
Lecture 36

IVP – ODE $\quad \dfrac{dy}{dt} = f(t, y(t))$

- Single-step method

  ↓

  use only previous time step to compute solution at current time step



- multi-step method

- Single step method

  - Euler's method
    - → forward Euler method / explicit Euler method
    - → backward Euler method / implicit Euler method

  - Heun's method

  - Mid point method

  - Runge-Kutta method

- Euler's method

- Forward Euler Method

$$\frac{dy(t)}{dt} = f(t, y(t)), \quad y(0) = y_0$$

examples
- $f = a\,y(t)$
- $f = g(t) + a\,y(t)^2$

$t_1, \; t_2, \; \ldots, \; t_N$

$\Delta t_1 = t_2 - t_1$

$\Delta t_2 = t_3 - t_2$

$\vdots$

$\Delta t_i = t_{i+1} - t_i$

$$\frac{dy}{dt}(t_i) \approx \frac{y(t_{i+1}) - y(t_i)}{\Delta t_i}$$

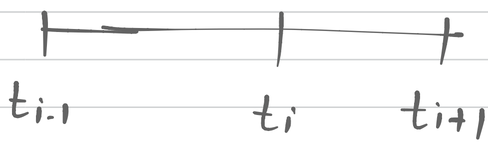
$$\frac{dy}{dt}(t_2) = f(t_2, y(t_2))$$

$\vdots$

$$\frac{dy}{dt}(t_i) = f(t_i, y(t_i))$$

$\downarrow$

$$\frac{y(t_{i+1}) - y(t_i)}{\Delta t_i} \approx f(t_i, y(t_i))$$



$$\Rightarrow \boxed{y(t_{i+1}) = y(t_i) + \Delta t_i\, f(t_i, y(t_i))}$$

• Backward Euler Method

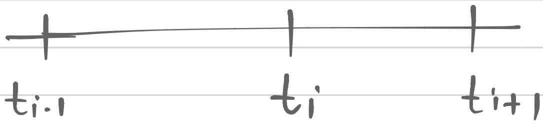
$$\frac{dy}{dt}(t_2) = f(t_2, y(t_2))$$

$$\frac{dy}{dt}(t_3) = f(t_3, y(t_3))$$

$$\vdots$$

$$\frac{dy}{dt}(t_i) = f(t_i, y(t_i))$$

$\Rightarrow$



$$\frac{dy}{dt}(t_i) \approx \frac{y(t_i) - y(t_{i-1})}{\Delta t_{i-1}}$$

$$\Rightarrow \frac{y(t_i) - y(t_{i-1})}{\Delta t_{i-1}} = f(t_i, y(t_i))$$

$y_i := y(t_i)$

$f_i := f(t_i, y_i)$

$$\Rightarrow \boxed{y_i = \underbrace{y_{i-1} + \Delta t_{i-1} f(t_i, y_i)}}$$

$$g(y_i) := y_{i-1} + \Delta t_{i-1} f(t_i, y_i)$$

$\Rightarrow$ $\boxed{y_i = g(y_i)}$ solve for y_i

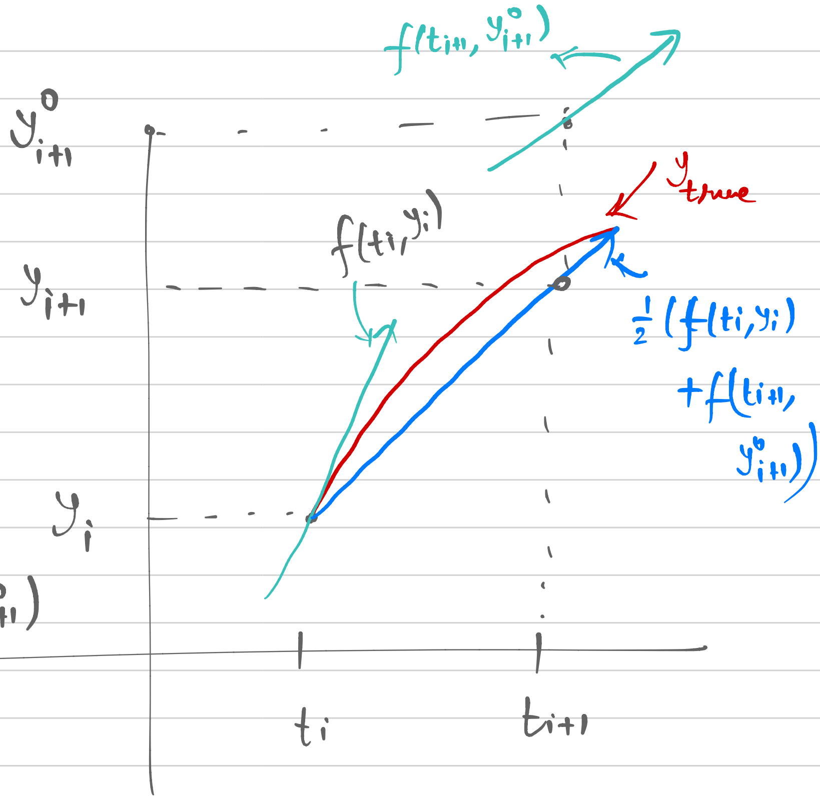
• Heun's method

<u>Given</u> t_i, y_i

- $y^0_{i+1} = y_i + (\Delta t_i) f(t_i, y_i)$

- $f(t_{i+1}, y^0_{i+1})$

- $\overline{\dfrac{dy}{dt}}(t_i) = \dfrac{f(t_i, y_i) + f(t_{i+1}, y^0_{i+1})}{2}$



$\Rightarrow \quad \dfrac{dy}{dt}(t_i) \approx \overline{\dfrac{dy}{dt}}(t_i)$

$\Rightarrow \quad \dfrac{y_{i+1} - y_i}{\Delta t_i} = \dfrac{1}{2}\left(f(t_i, y_i) + f(t_{i+1}, y^0_{i+1}) \right)$

$\Rightarrow \quad y_{i+1} = y_i + \dfrac{\Delta t_i}{2}\left(f(t_i, y_i) + f(t_{i+1}, y^0_{i+1}) \right)$

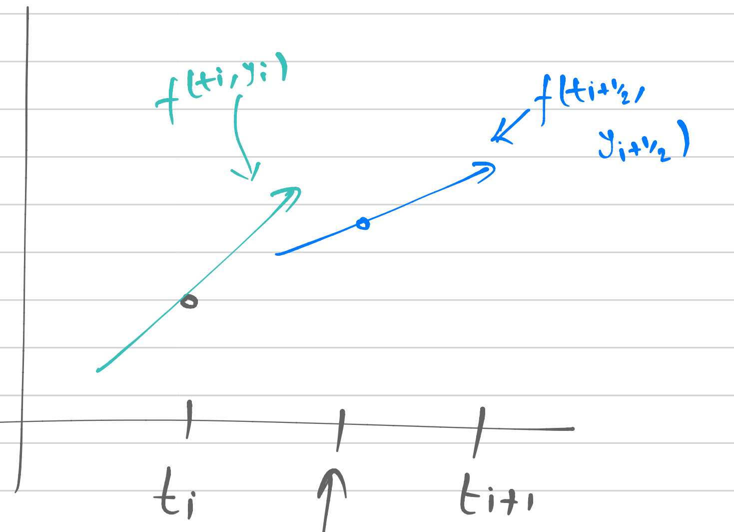
<u>Essentially</u> Given (t_i, y_i)

(i) $\quad y^0_{i+1} = y_i + \Delta t_i \, f(t_i, y_i)$

(ii) $\quad y_{i+1} = y_i + \dfrac{\Delta t_i}{2}\left(f(t_i, y_i) + f(t_{i+1}, y^0_{i+1}) \right)$

• Midpoint method

Given (t_i, y_i), we want to compute (t_{i+1}, y_{i+1}),



• $\underbrace{y_{i+1/2} = y_i + \frac{\Delta t_i}{2} f(t_i, y_i)}$

$$\frac{y(t_{i+1/2}) - y(t_i)}{\Delta t_i / 2} \approx f(t_i, y(t_i))$$

• $t_i + \frac{\Delta t_i}{2} = t_{i+1/2}$

• $y_{i+1/2} = y(t_{i+1/2})$

• $$\frac{dy}{dt}(t_i) = f(t_{i+1/2}, y_{i+1/2})$$

(Compare this $\frac{dy}{dt}(t_i) = f(t_i, y_i)$

$$\Rightarrow \frac{y_{i+1} - y_i}{\Delta t_i} = f(t_{i+1/2}, y_{i+1/2})$$

$$\Rightarrow y_{i+1} = y_i + \Delta t_i \, f(t_{i+1/2}, y_{i+1/2}$$

Essentially

(i) $y_{i+1/2} = y_i + \frac{\Delta t_i}{2} f(t_i, y_i)$

(ii) $y_{i+1} = y_i + \Delta t_i \, f(t_{i+1/2}, y_{i+1/2})$