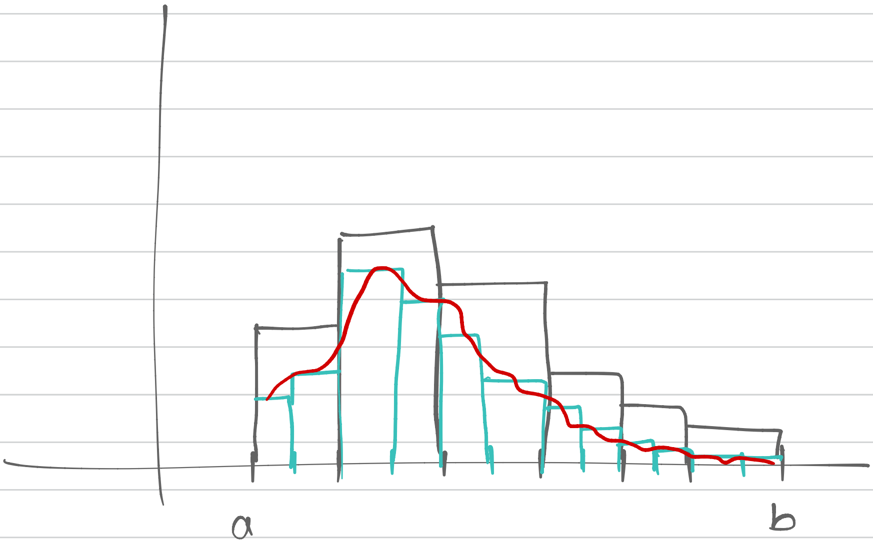
Lecture 25



Probability distribution function

$$f : [a, b] \to [0, \infty)$$

(i) $f(x) \geq 0$, $f(x) \leq 1$

(ii) $\displaystyle\int_a^b f(x)\,dx = 1$

If you have a function $g : [a, b] \to [0, \infty)$

$$f(x) = \frac{g(x)}{\int_a^b g(x)\,dx} \Rightarrow \int_a^b f(x)\,dx = 1$$

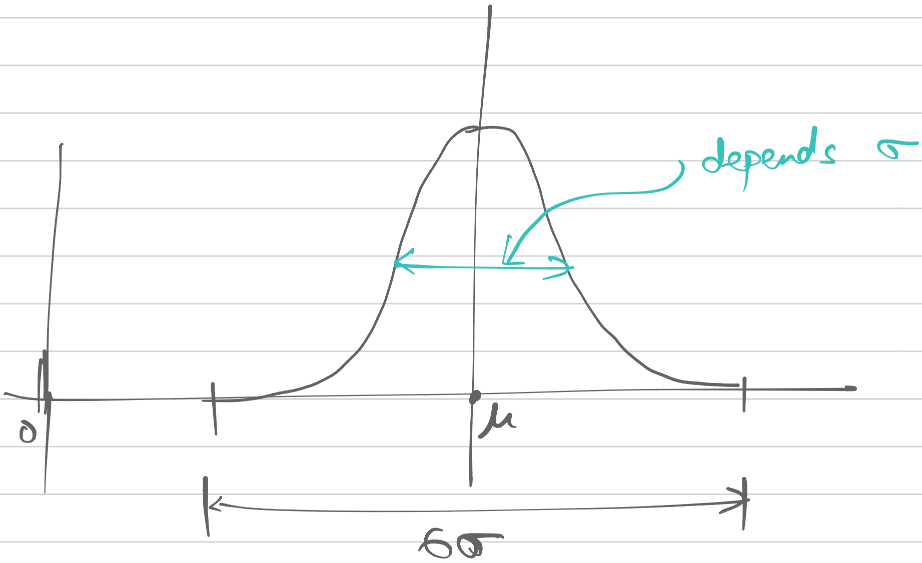
scalar numb

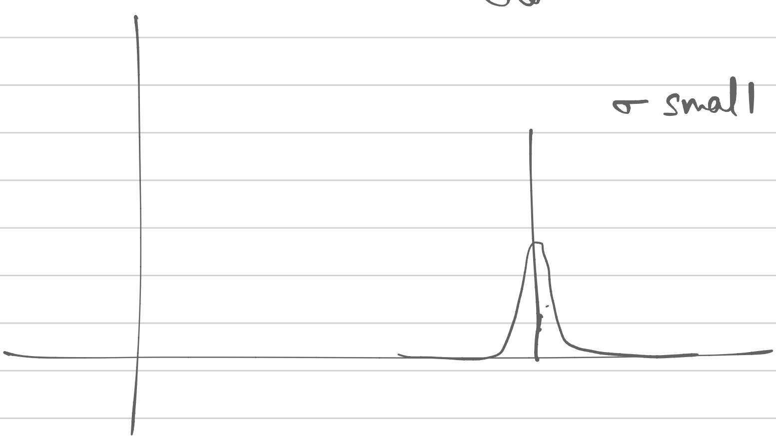
Gaussian distribution function

$$G(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

σ = standard deviation

μ = mean





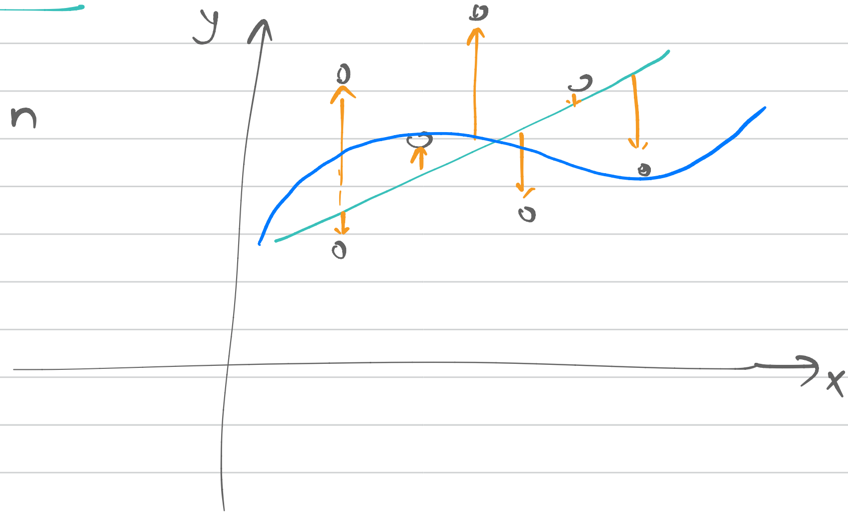
- Linear regression

data

(x_i, y_i), $i = 1, 2, \ldots, n$

fit straight line

$y = y(x) = a_0 + a_1 x$



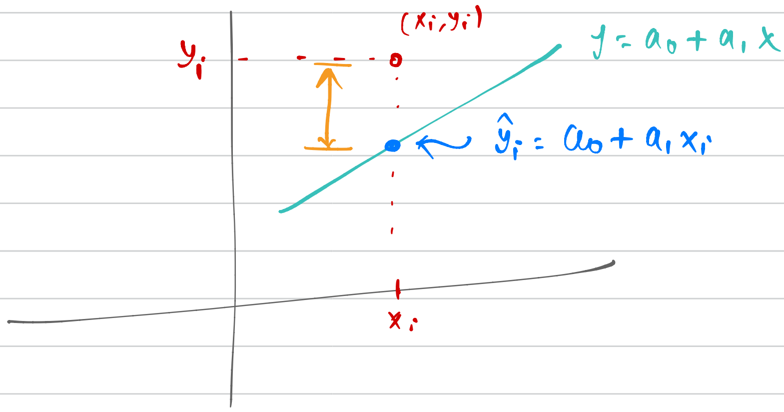
Examples

(i)	$y(x) = a_0 + a_1 x$	linear curve	linear regression
(ii)	$y(x) = a_0 + a_1 x + a_2 x^2$	quadratic curve	linear regression
(iii)	$y(x) = a_0 + \sin(a_1 x) + \cos(a_2 x)^2$	nonlinear curve	nonlinear regression

$\alpha_i = y_i - \hat{y}_i$

$\beta_i = |y_i - \hat{y}_i|$

$\gamma_i = |y_i - \hat{y}_i|^2$

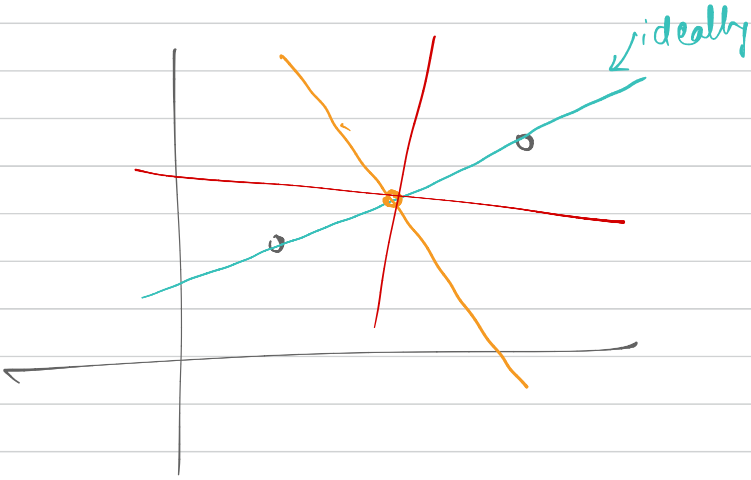


Total error

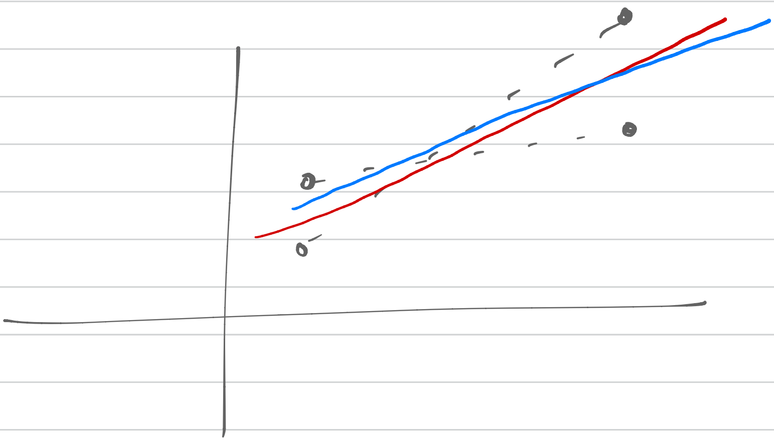
$$E = \sum_{i=1}^{n} \text{distance } (x_i, y_i) \text{ and } (x_i, \hat{y}_i)$$

$\times$ $E_\alpha = \sum_{i=1}^{n} \alpha_i$, $\times$ $E_\beta = \sum_{i=1}^{n} \beta_i$, $\boxed{E_\gamma = \sum_{i=1}^{n} \gamma_i}$

Two data (x_1, y_1) (x_2, y_2)



using E_α errors



E_β errors

- least square method (straight line)

$$E(a_0, a_1) = E = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \quad , \hat{y}_i = a_0 + a_1 x_i$$

(x_i, y_i) data

find a_0, a_1 that minimizes $E(a_0, a_1)$

→ at minimum a_0 and a_1

$$\frac{\partial E}{\partial a_0} = 0$$

$$\frac{\partial E}{\partial a_1} = 0$$

$$\frac{\partial E}{\partial a_0} = \frac{\partial}{\partial a_0} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \quad , \quad \hat{y}_i = a_0 + a_1 x_i$$

$$\frac{\partial \hat{y}_i}{\partial a_0} = 1$$

$$\frac{\partial \hat{y}_i}{\partial a_1} = x_i$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial a_0} (\hat{y}_i - y_i)^2$$

$$= \sum_{i=1}^{n} 2(\hat{y}_i - y_i) \left(\frac{\partial \hat{y}_i}{\partial a_0} - \cancel{\frac{\partial y_i}{\partial a_0}}^{\,0} \right)$$

$$= \sum_{i=1}^{n} 2(\hat{y}_i - y_i)(1 - 0)$$

$$\Rightarrow \frac{\partial E}{\partial a_0} = 2 \sum_{i=1}^{n} (a_0 + a_1 x_i - y_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} a_0 + \sum_{i=1}^{n} a_1 x_i - \sum_{i=1}^{n} y_i = 0$$

Note that

- $\sum_{i=1}^{n} a_0 = a_0 \sum_{i=1}^{n} 1 = n a_0$
- $\sum_{i=1}^{n} a_1 x_i = a_1 \sum_{i=1}^{n} x_i \rightarrow n\bar{x}$, $\quad \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$
- $\sum_{i=1}^{n} y_i = n\bar{y}$, $\quad \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$

$$\Rightarrow n a_0 + n\bar{x} a_1 - n\bar{y} = 0$$

$$\Rightarrow \boxed{a_0 + \bar{x} a_1 = \bar{y}} \longrightarrow \frac{\partial E}{\partial a_0} = 0$$

$$\frac{\partial E}{\partial a_1} = \sum_{i=1}^{n} 2(\hat{y}_i - y_i) \left(\frac{\partial \hat{y}_i}{\partial a_1} \right) = \sum_{i=1}^{n} 2(a_0 + a_1 x_i - y_i) x_i$$

$$= 0$$

$$\Rightarrow \sum_{i=1}^{n} a_0 x_i + \sum_{i=1}^{n} a_1 x_i^2 - \sum_{i=1}^{n} y_i x_i = 0$$

$$\Rightarrow a_0 n\bar{x} + a_1 \left(\sum_{i=1}^{n} x_i^2 \right) - \sum_{i=1}^{n} y_i x_i = 0$$

$$\Rightarrow \boxed{a_0 n\bar{x} + a_1 \left(\sum_{i=1}^{n} x_i^2 \right) = \sum_{i=1}^{n} x_i y_i}$$

$$J a = b$$

$$J = \begin{bmatrix} 1 & \bar{x} \\ n\bar{x} & \sum_{i=1}^{n} x_i^2 \end{bmatrix}, \quad a = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \quad b = \begin{bmatrix} n\bar{y} \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

fitting quadratic curve

$$y = a_0 + a_1 x + a_2 x^2$$