Lecture 14

Methods to solve $Ax = b$

(i) graphical method

(ii) direct method

(i) inverse $x = A^{-1}b$

(ii) Cramer's rule

(iii) Gauss-elimination

Cramer's rule :

$\underbrace{\text{2 equations}}_{\text{2 rows}}$ $\underbrace{\text{2 unknowns}}_{\text{2 columns}}$

① $a_{11}x_1 + a_{12}x_2 = b_1$

② $a_{21}x_1 + a_{22}x_2 = b_2$

② $- \frac{a_{21}}{a_{11}}$ ①

$$\left(a_{21} - \frac{a_{21}}{\cancel{a_{11}}}\cancel{a_{11}}\right)x_1 + \left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 = b_2 - \frac{a_{21}}{a_{11}}b_1$$

$$\Rightarrow \left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 = b_2 - \frac{a_{21}}{a_{11}}b_1$$

$$\Rightarrow (a_{11}a_{22} - a_{21}a_{12})x_2 = a_{11}b_2 - a_{21}b_1$$

$$\Rightarrow \boxed{x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}}$$

Substitute x_2 into ①

$$a_{11}x_1 + a_{12}\left(\frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}\right) = b_1$$

$$\Rightarrow \quad x_1 = \frac{1}{a_{11}}\left[b_1 - \frac{a_{12}\,(a_{11}b_2 - a_{21}b_1)}{a_{11}a_{22} - a_{21}a_{12}} \right]$$

$$\Rightarrow \quad x_1 = \frac{1}{a_{11}}\left[\frac{b_1 a_{11} a_{22} - b_1 \cancel{a_{21}} a_{12} - a_{12} a_{11} b_2 + \cancel{a_{12}} a_{21} b_1}{a_{11}a_{22} - a_{21}a_{12}} \right]$$

$$\Rightarrow \quad \boxed{x_1 = \frac{\overbrace{a_{22}b_1 - a_{12}b_2}^{\to\, D_1}}{\underbrace{a_{11}a_{22} - a_{21}a_{12}}_{\to\, D}}}$$

$$\boxed{x_2 = \frac{\overbrace{a_{11}b_2 - a_{21}b_1}^{\to\, D_2}}{\underbrace{a_{11}a_{22} - a_{21}a_{12}}_{\to\, D}}}$$

$$D = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{21}a_{12}$$

$$D_1 = \det\left(\begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}\right), \quad D_2 = \det\left(\begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}\right)$$

$$x_1 = \frac{\det\left(\begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}\right)}{\det\left(\begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix}\right)}, \quad x_2 = \frac{\det\left(\begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix}\right)}$$

3 equations 3 unknowns

(1) $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

(2) $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

(3) $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

$$x_1 = \frac{\det\left(\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}\right)}{\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = D} \ldots$$

$$x_2 = \frac{\det\left(\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}\right)}{D}$$

$$x_3 = \frac{\det\left(\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}\right)}{D}$$

for general n equation, n unknown

$$x_i = \frac{\det\left(\begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & & b_2 & & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & & b_n & & a_{nn} \end{bmatrix}\right)}{\det(A)}$$

Gauss elimination method

two step method

1. Forward - elimination
2. Backward substitution

Forward elimination

$$\text{(1)}\quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\text{(2)}\quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$\text{(n)}\quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

step 1 : modify (2), (3), ... (n) equations such that it does not have "x_1" term

$$\text{(2)} - \frac{a_{21}}{a_{11}}\text{(1)}, \quad \text{(3)} - \frac{a_{31}}{a_{11}}\text{(1)}, \quad \ldots, \quad \text{(n)} - \frac{a_{n1}}{a_{11}}\text{(1)}$$

$$\text{(1)}\quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\text{(2)}\quad 0x_1 + \left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \cdots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$
$$\vdots$$
$$\text{(n)}\quad 0x_1 + \left(a_{n2} - \frac{a_{n1}}{a_{11}}a_{12}\right)x_2 + \cdots + \left(a_{nn} - \frac{a_{n1}}{a_{11}}a_{1n}\right)x_n = b_n - \frac{a_{n1}}{a_{11}}b_1$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \underbrace{a_{22} - \frac{a_{21}}{a_{11}}a_{12}}_{a_{22}^{(1)}} & \cdots & \underbrace{a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}}_{a_{2n}^{(1)}} \\ 0 & \underbrace{a_{n2} - \frac{a_{n1}}{a_{11}}a_{12}}_{a_{n2}^{(1)}} & & \underbrace{a_{nn} - \frac{a_{n1}}{a_{11}}a_{1n}}_{a_{nn}^{(1)}} \end{bmatrix} x = \begin{bmatrix} b_1 \\ \underbrace{b_2 - \frac{a_{21}}{a_{11}}b_1}_{b_2^{(1)}} \\ \vdots \\ \underbrace{b_n - \frac{a_{n1}}{a_{11}}b_1}_{b_n^{(1)}} \end{bmatrix}$$

$$① \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$② \quad 0x_1 + a_{22}^{(1)}x_2 + \cdots + a_{2n}^{(1)}x_n = b_2^{(1)}$$

$$\vdots$$

$$ⓝ \quad 0x_1 + a_{n2}^{(1)}x_2 + \cdots + a_{nn}^{(1)}x_n = b_n^{(1)}$$

$$③ - \frac{a_{32}^{(1)}}{a_{22}^{(1)}}② , \quad \ldots \quad , ⓝ - \frac{a_{n2}^{(1)}}{a_{22}^{(1)}}②$$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}^{(1)}x_2 + \cdots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\
0x_2 + \cdots + a_{3n}^{(2)}x_n &= b_3^{(2)} \\
\\
0x_2 + \cdots + a_{nn}^{(2)}x_n &= b_n^{(2)}
\end{aligned}$$

after $(n-1)$ step

$$① \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$② \quad a_{22}^{(1)}x_2 + \cdots + a_{2n}^{(1)}x_n = b_2^{(1)}$$

$$③ \quad 0 + a_{33}^{(2)}x_3 + \cdots + a_{3n}^{(2)}x_n = b_3^{(2)}$$

$$\vdots$$

$$0x_1 + 0x_2 + \cdots + 0x_{n-1} + a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

$$\begin{bmatrix} a_{11} & a_{12} & & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & & \cdots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ 0 & 0 & & 0 & a_{nn}^{(n-1)} \end{bmatrix} X = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Backward substitution :

(n)

$$a_{nn}^{(n-1)} x_n = b_n^{(n-1)}$$

$$\Rightarrow \boxed{x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}}$$

(n-1)

$$a_{(n-1)(n-1)}^{(n-2)} x_{n-1} + a_{(n-1)n}^{(n-2)} x_n = b_{n-1}^{(n-2)}$$

$$\Rightarrow a_{(n-1)(n-1)}^{(n-2)} x_{n-1} + a_{(n-1)n}^{(n-2)} x_n = b_{n-1}^{(n-2)}$$

$$\Rightarrow \boxed{x_{n-1} = \frac{b_{n-1}^{(n-2)} - a_{(n-1)n}^{(n-2)} x_n}{a_{(n-1)(n-1)}^{(n-2)}}}$$

(n-2) $\quad i = n-2, \quad j = n-1, \quad k = n, \quad \alpha = n-3$

$$a_{ii}^{(\alpha)} x_i + a_{ij}^{(\alpha)} x_j + a_{ik}^{(\alpha)} x_k = b_i^{(\alpha)}$$

$$x_i = \frac{b_i^{(\alpha)} - a_{ij}^{(\alpha)} x_j - a_{ik}^{(\alpha)} x_k}{a_{ii}^{(\alpha)}}$$

$$\boxed{x_{n-2} = \frac{b_{(n-2)}^{(n-3)} - a_{(n-2)(n-1)}^{(n-3)} x_{n-1} - a_{(n-2)n}^{(n-3)} x_n}{a_{(n-2)(n-2)}^{(n-3)}}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i(i+1)}^{(i-1)} x_{i+1} - a_{i(i+2)}^{(i-1)} x_{i+2} \cdots - a_{in}^{(i-1)} x_n}{a_{ii}^{(i-1)}}$$

$$\boxed{x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}}$$