Lecture 9

Open methods

1. Fixed point iteration method
2. Newton-Raphson method
3. Secant method
4. Brent's method

$f(x_0) = 0$

Given a function $g : X \to (-\infty, \infty)$ find a point $x_0 \in X$ such that following holds

$$x_0 = g(x_0)$$

Fixed-point iteration method

Consider following function $f : X \to Y$

$$f(x) = x - g(x)$$

where g is another function $g : X \to Y$.

Roots of function f: find $x_0 \in X$ such that

$$f(x_0) = 0 \Rightarrow x_0 - g(x_0) = 0$$

$$\Rightarrow \boxed{x_0 = g(x_0)}$$

$$\Downarrow$$

find x_0 such that $x_0 = g(x_0)$.

For any function f: we can always have

$$f(x) = x - g(x)$$

by defining $\boxed{g(x) := x - f(x)}$ $\Rightarrow f(x_0) = 0$ $\Rightarrow x_0 = g(x_0)$

Thus for any function f: root problem can be written as "find x_0 such that $x_0 = g(x_0)$"

!!! Problem of finding x such that $x = g(x)$ is called fixed-point iteration problem

How to solve $x = g(x)$?

- Suppose x^0 is the initial guess
- then we find the next x by using $\boxed{x^1 = g(x^0)}$

$\vdots$

find the x at i^{th} iteration $\boxed{x^i = g(x^{i-1})}$

we perform this iteration until error $e_a = \dfrac{|x^i - x^{i-1}|}{|x^i|} \times 100\%$ is below our tolerance.

$\Rightarrow$ Easy to implement in MATLAB

$\Rightarrow$ However, we first need to study the properties of the iterative method $x^i = g(x^{i-1})$

$f(x)=0$

$\Rightarrow x=1$

Example 1 $f(x) = (x-1)^2$, $X = (-\infty, \infty)$, $Y = [0, \infty)$

Let $g(x) = x - f(x) = x - (x-1)^2$

Let initial guess is $x^0 = 0.5$

iteration 1 : $x^1 = g(x^0) = 0.5 - 0.25 = 0.25$

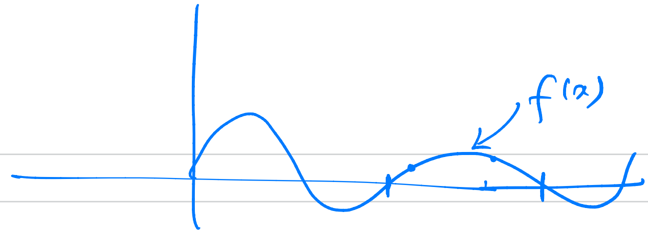
iteration 2 : $x^2 = g(x^1) = 0.25 - 0.5625 = -0.3125$

iteration 3 : $x^3 = g(x^2) = -0.3125 - (1.3125)^2 = -2.035$

iteration 4 : $x^4 = g(x^3) = -2.035 - (-2.035-1)^2 = -11.25$

$\vdots$

diverging



Let initial guess $x^0 = 1.1$

Then iteration 1 : $x^1 = g(x^0) = 1.1 - 0.01 = 1.09$

iteration 2 : $x^2 = g(x^1) = 1.09 - (0.09)^2 = 1.0819$

iteration 3 : $x^3 = g(x^2) = 1.0752$

⋮

Converging to $x_0 = 1$

Example 2 :

$f(x) = x - \cos(x)$, $X = (-\infty, \infty)$, $Y = (-\infty, \infty)$

Then $g(x) = x - f(x) = \cos(x)$

Initial guess : $x^0 = 0.5$

iter. 1 : $x^1 = g(x^0) = \cos(0.5) = 0.8776$

iter. 2 : $x^2 = g(x^1) = \cos(0.8776) = 0.639$

iter. 3 : $x^3 = \cos(0.639) = 0.803$

iter. 4 : $x^4 = 0.695$

iter. 5 : $x^6 = 0.768$

iter. 6 : $x^7 = 0.7193$

iter. 7 : $x^8 = 0.752$

⋮

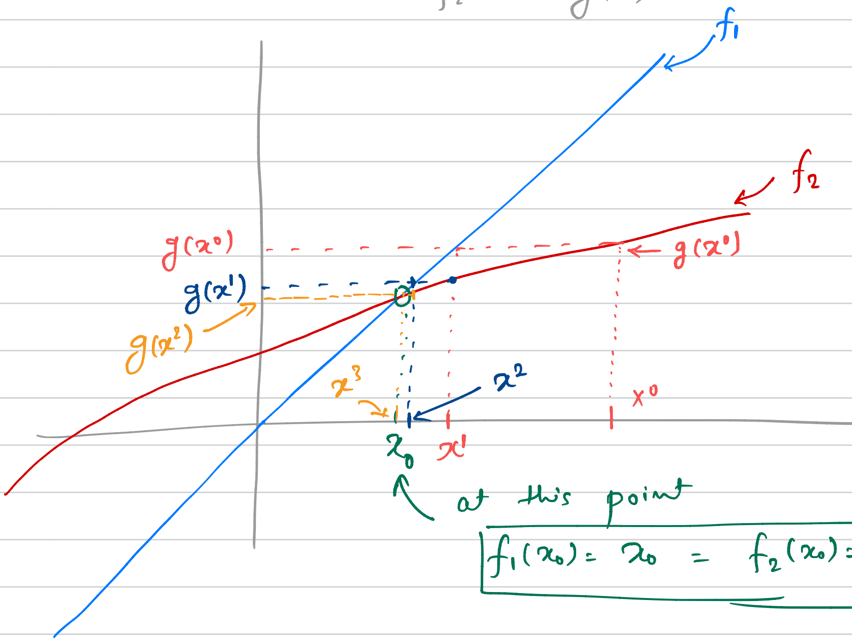
$x^{\vdots} = 0.7388$

Converging

To understand how fixed-point iteration works

let $f_1(x) = x$

$f_2(x) = g(x)$



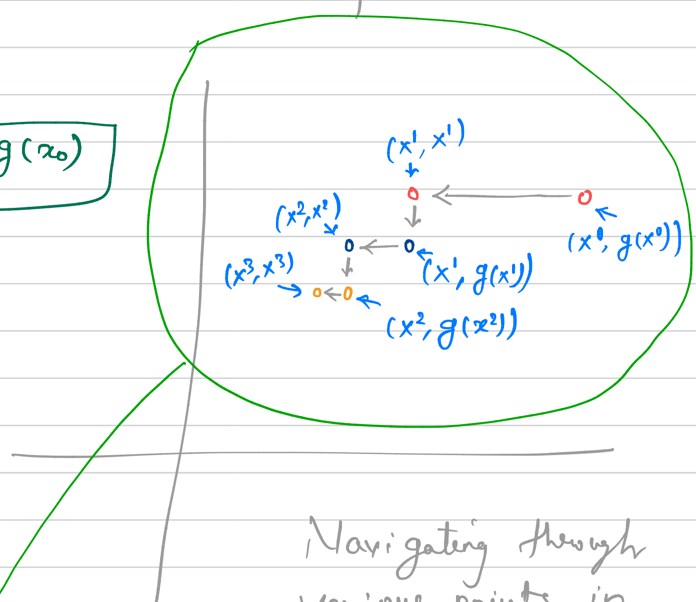
at this point

$$\boxed{f_1(x_0) = x_0 = f_2(x_0) = g(x_0)}$$

The solution of $x = g(x)$ problem is a point x_0 such that

$$f_1(x_0) = f_2(x_0)$$

I.e. point at which two functions intersect



Plot our iteration steps:

iter. 1 : $x^1 = g(x^0)$

iter 2 : $x^2 = g(x^1)$

iter 3 : $x^3 = g(x^2)$

⋮

Navigating through various points in fixed-point iteration method

Can we say more about this particular example ?

⇓

Given a point x^i, we compute $x^{i+1} = g(x^i)$

we see that

$$\left\{ \begin{array}{l} x^1 = g(x^0) < x^0 \\ x^2 = g(x^1) < x^1 \\ x^3 = g(x^2) < x^2 \\ \vdots \end{array} \right.$$

Generally, for any $x > x_0$

$g(x) < x$

where x_0 is the true solution of $x = g(x)$

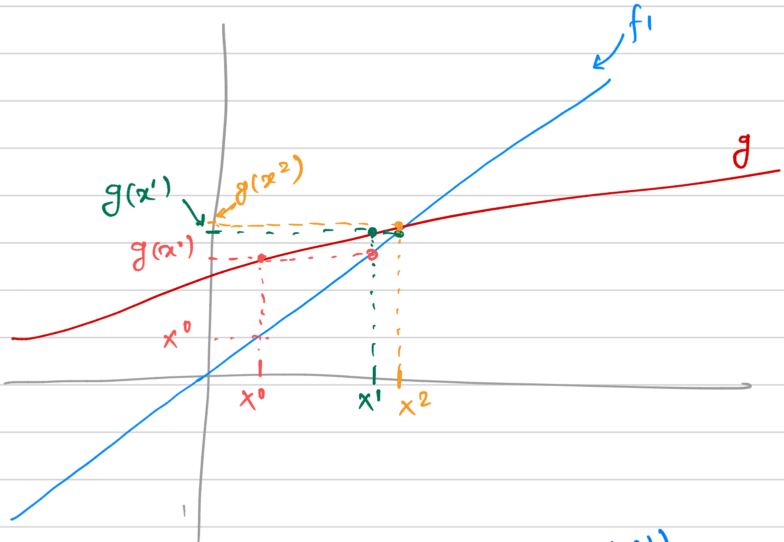
Previously, we considered a function g such that

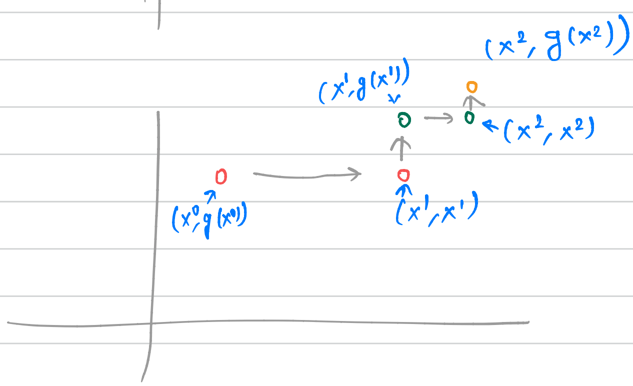
$$g(x) < x \qquad \text{for any } x > x_0$$

x_0 ← true solution of $x = g(x)$

→ for such a function, we see that in each iteration we got closer to true solution x_0.

Let us see now the case when x^0 (initial guess) is on the left side of true solution:





⇒ In this case also we see that in each iteration we are getting closer to true solution x_0

⇓

"we observe"

$$x^1 = g(x^0) > x^0$$

$$x^2 = g(x^1) > x^1$$

$$x^3 = g(x^2) > x^2$$

↓

for $x < x_0$ (where x_0 is the true solution), we have

$$g(x) > x$$

initial guess

true solution

Thus if

(i) we start from right side of x_0, i.e. $x^0 > x_0$

we want $g(x) < x$ for any $x > x_0$,

So that successive iterations will reduce

x^i trying to get closer to x_0

I.e. need $g(x) < x$ for $x > x_0$ so

that we get

$$x^0 > x^1 > x^2 > x^3 \cdots > x_0$$

(ii) We start from left side of x_0, i.e. $x^0 < x_0$,

then we want $g(x) > x$ for any $x < x_0$,

So that successive iterations will increase x^i

taking it closer to x_0

I.e. need $g(x) > x$ for $x < x_0$ so

$$x^0 < x^1 < x^2 < x^3 < \cdot \cdot < x_0$$

$\Downarrow$

What happens when g does not have this property?

Is it still possible to converge to x_0?

Error in fixed point iteration method

let x_0 is such that $x_0 = g(x_0)$ (So x_0 is the true solution)

Let E_t^i := true error at iteration i

$= x^i - x_0$

Since $x^i = g(x^{i-1})$ ← our iteration method!

$\Rightarrow E_t^i = x^i - x_0$

(1) $= g(x^{i-1}) - x_0$

$$\Rightarrow \boxed{E_t^i = g(x^{i-1}) - g(x_0)}$$

(∵ x_0 is true solution so $x_0 = g(x_0)$)

We know from Taylor's series expansion

$$f(x) = f(y) + \frac{df}{dy}(y)(x-y) + \frac{1}{2!}\frac{d^2f}{dy^2}(y)(x-y)^2$$

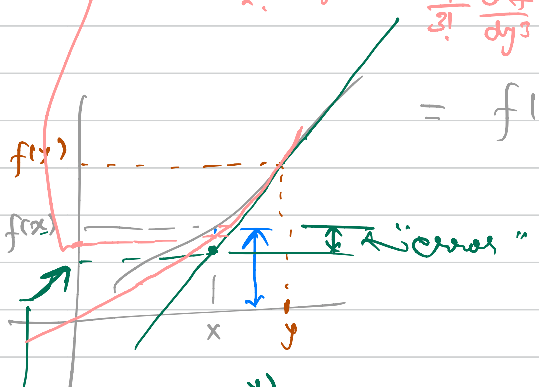
$$+ \dots + \frac{1}{n!}\frac{d^nf}{dy^n}(y)(x-y)^n + \dots$$

$2! = 2$

$f(y) + \frac{df}{dy}(y)(x-y)$

$+ \frac{1}{2!}\frac{d^2f}{dy^2}(y)(x-y)^2$

$\frac{1}{3!}\frac{d^3f}{dy^3}(z)(x-y)^3$



$f(y) + \frac{df}{dy}(y)(x-y)$

$$= f(y) + \frac{df}{dy}(y)(x-y) + \frac{1}{2!}\frac{d^2f}{dy^2}(y)(x-y)^2$$

$$+ \dots + \frac{1}{n!}\frac{d^nf}{dy^n}(z)(x-y)^n$$

there exists $z \in X$

and z depends on choice of x, y, n

$$\boxed{f(x) = f(y) + \frac{df(z)}{dy}(x-y)}$$

there exist $z \in X$

z depends on x and y

$f(y) + \frac{df}{dy}(y)(x-y)$

$+ \frac{1}{2!}\frac{d^2f}{dy^2}(z)(x-y)^2$

We can write

(2) $$\boxed{g(x^{i-1}) = g(x_0) + \frac{dg(z)}{dy}(x^{i-1} - x_0)}$$

z is not known and generally z will depend on x^{i-1} and x_0

Thus combining (1) and (2)

$$E_t^i = \frac{dg(z)}{dy}\overbrace{(x^{i-1} - x_0)}^{E_t^{i-1}}$$

(3) $$\Rightarrow \boxed{E_t^i = \frac{dg(z)}{dy} E_t^{i-1}} \Rightarrow \frac{|E_t^i|}{|E_t^{i-1}|} \leq \left|\frac{dg(z)}{dy}\right| \leq M$$

Suppose such a number M exist

$$\frac{|E_t^i|}{|E_t^{i-1}|} < 1 \Rightarrow |E_t^i| < |E_t^{i-1}|$$

" at any point $x \in X$, $\left|\frac{dg(x)}{dy}\right| < 1$ "

Equation (3) is very important result and provides mathematical reasoning as to when error will decrease in successive iterations and when it will increase

$\Downarrow$

when method will converge and when it will diverge

We want x^i to get closer and closer to x_0 with increasing i

I.e. $$|E_t^1| > |E_t^2| > |E_t^3| > \dots > |E_t^i| > \dots$$

OR $$1 > \frac{|E_t^2|}{|E_t^1|},\quad 1 > \frac{|E_t^3|}{|E_t^2|},\quad \dots\quad 1 > \frac{|E_t^i|}{|E_t^{i-1}|}$$

Since $\dfrac{|\epsilon_t^{i}|}{|\epsilon_t^{i-1}|} \leq \left|\dfrac{dg(z)}{dy}\right|$ for any $z \in X$

Convergence is guaranteed if $\left|\dfrac{dg(x)}{dy}\right| < 1$ for all points $x \in X$

$$\Downarrow$$

$$\frac{|\epsilon_t^{i}|}{|\epsilon_t^{i-1}|} < 1$$

$$\Downarrow$$

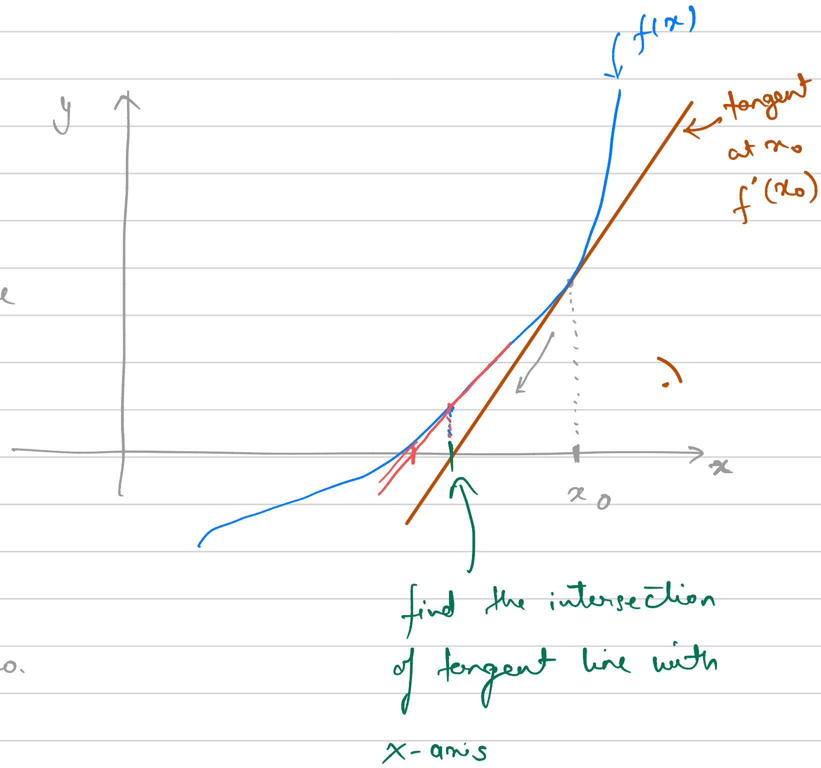
Thus fixed-point iteration converges surely if slope of function g at any point $x \in X$ is below 1

Newton - Raphson method

— let's look at method graphically

— consider a initial guess

x_0



— find the equation for tangent line at x_0

$y = mx + c$

where m = slope

c = height of line at $x = 0$.

(i) For tangent line, slope $= m = f'(x_0)$

& $\quad y = x f'(x_0) + c$

(ii) Tangent line passes through point $(x_0, f(x_0))$

x-coordinate — y-coordinate

$\Rightarrow \quad f(x_0) = x_0 f'(x_0) + c$

$\Rightarrow \quad c = f(x_0) - x_0 f'(x_0)$

Thus the equation of tangent line is

$$y = x f'(x_0) + f(x_0) - x_0 f'(x_0)$$

$$\Rightarrow y(x) = (x - x_0) f'(x_0) + f(x_0)$$

— Find $\bar{x}$ at which line intersects x-axis (x-axis means $y = 0$)

So $y(\bar{x}) = 0$

$$\Rightarrow (\bar{x} - x_0) f'(x_0) + f(x_0) = 0$$

$$\Rightarrow \bar{x} - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow \boxed{\bar{x} = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

— So if x_0 is initial guess, we will take $\bar{x}$ as next guess

set $\boxed{x_1 = \bar{x} = x_0 - \frac{f(x_0)}{f'(x_0)}}$

— Now we have x_1 guess and we use same procedure to find x_2 guess :

(i) Create a tangent line passing through $(x_1, f(x_1))$ with slope $f'(x_1)$ $\Rightarrow$ $y(x) = (x - x_1) f'(x_1) + f(x_1)$

(ii) find x_2 s.t. $y(x_2) = 0$

$$\Rightarrow \boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}$$