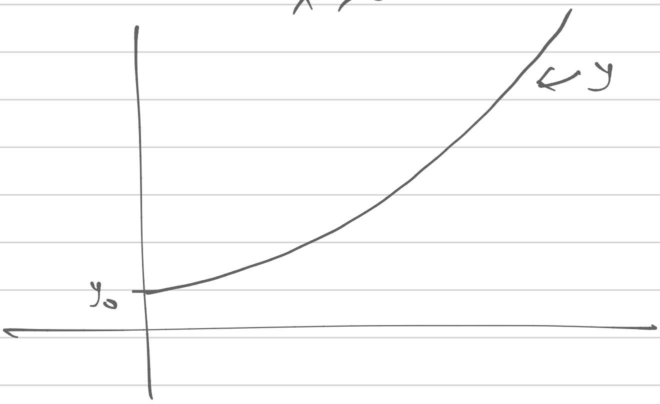
Lecture 39

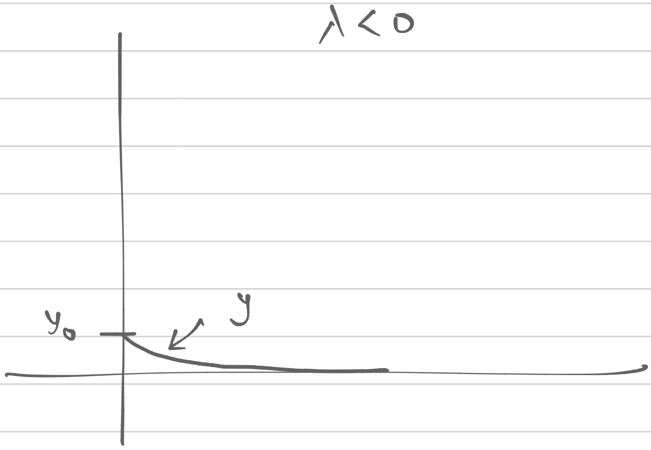
- Errors (consistency error) : discretization errors
- Stability : solution is convergent / divergent

$$\frac{dy}{dt} = \lambda y \quad , \quad y(0) = y_0 \quad \Rightarrow \quad y(t) = y_0 e^{\lambda t}$$

$\lambda > 0$



$\lambda < 0$



$$y_{i+1} = y_i + \lambda y_i \Delta t$$ forward Euler

$$\Rightarrow y_{i+1} = y_i (1 + \lambda \Delta t) \longrightarrow y_i = y_{i-1} (1 + \lambda \Delta t)$$

$$= y_{i-1} (1 + \lambda \Delta t)^2$$

$$= y_{i-2} (1 + \lambda \Delta t)^3$$

$$\Rightarrow \boxed{y_{i+1} = y_0 (1 + \lambda \Delta t)^{i+1}}$$

if $i \longrightarrow \infty$

$T = 1, \quad \Delta t = 10^{-6}$

$i = 1, 2, \ldots, 10^6$

$a = 1 + \lambda \Delta t,$ $a^i \longrightarrow \infty$ if $a > 1$, diverging (unstable) diverging

$a^i \longrightarrow -\infty$ if i odd
$a^i \longrightarrow \infty$ if i even
if $a < -1$

$a^i \longrightarrow 0$
if $-1 < a < 1$

converging (stable)

$a^i \longrightarrow 1$ or -1
if $a = -1$

$a^i \longrightarrow 1$
if $a = 1$,

(1) $\Delta t > 0$

(2) if $\lambda > 0$, $a = 1 + \lambda \Delta t > 1$

(3) if $\lambda < 0$, $a = 1 + \lambda \Delta t$

we want Δt s.t.

$$-1 \leq a \leq 1 \Rightarrow -1 \leq 1 + \lambda \Delta t \leq 1$$

$$-1 \leq 1 + \lambda \Delta t$$

$$1 + \lambda \Delta t \leq 1$$

trivial

$$\Rightarrow 0 \leq 2 + \lambda \Delta t$$

$$\Rightarrow (-\lambda) \Delta t \leq 2$$

$$\Rightarrow \boxed{\Delta t \leq \frac{2}{-\lambda}}$$

as long as

$\Delta t \leq -\frac{2}{\lambda}$ (with $\lambda < 0$)

forward Euler is stable!

$ab \leq 1$
if $a < 0$
$b \geq \frac{1}{a}$
if $a > 0$
$b \leq \frac{1}{a}$

System of 1^{st} order ODEs

$$\frac{dy}{dt} = A y, \quad y = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \quad y(0) = \begin{bmatrix} y_1^0 \\ y_2^0 \\ \vdots \\ y_n^0 \end{bmatrix}$$

A = $n \times n$ matrix

(1) forward Euler:

look at i^{th} eq

$$\frac{d y_i(t)}{dt} = \sum_{j=1}^{n} a_{ij} \, y_j(t)$$

$$\Rightarrow \frac{y_i(t_{k+1}) - y_i(t_k)}{\Delta t} = \sum_{j=1}^{n} a_{ij} \, y_j(t_k)$$

$$\Rightarrow \boxed{y_i(t_{k+1}) = y_i(t_k) + \Delta t \sum_{j=1}^{n} a_{ij} \, y_j(t_k)}$$

$$\boxed{y(t_{k+1}) = y(t_k) + \Delta t \, A y(t_k)}$$

discretization of vector equation

$$\Downarrow$$

$$\boxed{y(t_{k+1}) = (I + \Delta t A) \, y(t_k)}$$

I is identity matrix

(ii) Backward Euler

pick i^{th} equation

$$\frac{dy_i(t)}{dt} = \sum_{j=1}^{n} a_{ij}\, y_j(t)$$

$$\frac{y_i(t_{k+1}) - y_i(t_k)}{\Delta t} = \sum_{j=1}^{n} a_{ij}\, y_j(t_{k+1})$$

$$\Rightarrow \quad y_i(t_{k+1}) = y_i(t_k) + \Delta t \sum_{j=1}^{n} a_{ij}\, y_j(t_{k+1})$$

$$\Downarrow$$

$$\boxed{y(t_{k+1}) = y(t_k) + \Delta t\, A\, y(t_{k+1})}$$

$$\Downarrow$$

$$\Rightarrow \boxed{(I - \Delta t A)\, y(t_{k+1}) = y(t_k)}$$

define $J = I - \Delta t A$, $x = y(t_{k+1})$, $b = y(t_k)$

$$\Rightarrow \boxed{Jx = b}$$

Coming back to scalar ODE $\frac{dy}{dt} = \lambda y$

$$y_{i+1} = y_i + \lambda \Delta t \, y_{i+1} \qquad \text{(backward Euler)}$$

$$\Rightarrow (1 - \lambda \Delta t) \, y_{i+1} = y_i$$

$$\Rightarrow y_{i+1} = \frac{y_i}{1 - \lambda \Delta t}$$

$$= \frac{y_{i-1}}{(1 - \lambda \Delta t)^2} = \cdots = \frac{y_0}{(1 - \lambda \Delta t)^{i+1}}$$

$$\Rightarrow y_{i+1} = \frac{y_0}{(1 - \lambda \Delta t)^{i+1}} =$$

Points: (i) $\Delta t > 0$

(ii) for stable ODE, $\lambda < 0$

$$a = \frac{1}{1 - \lambda \Delta t} \Rightarrow y_{i+1} = a^{i+1} y_0$$

if $i \to \infty$, $a^i \to 0$

for any Δt

because $0 < a < 1$

Coming back to system of ODEs

- forward Euler $\quad y_{i+1} = (I + \Delta t A)\, y_i$

$$\Rightarrow y_{i+1} = \underbrace{(I + \Delta t A)}^{i+1} y_0$$

$$J = I + \Delta t A$$

$$y_{i+1} = J^{i+1} y_0$$

Spectral radius of $J \leq 1$ for this scheme to be stable.

$$\max(\text{abs}(\lambda_1, \lambda_2, \ldots, \lambda_n))$$

λ_i are eigenvalues of $J = I + \Delta t A$