Lecture 38

- Single step methods

ODE: $\frac{dy}{dt} = f(t, y)$, $y(0) = y_0$

notation: $y_i = y(t_i)$, $f_i = f(t_i, y_i)$

(1.) Forward Euler Method

$$y_{i+1} = y_i + \Delta t\, f_i$$

(2.) Backward Euler Method

$$y_{i+1} = y_i + \Delta t\, f_{i+1}$$

(3.) Heun's Method (Trapezoidal Method)

$$y_{i+1}^{0} = y_i + \Delta t\, f_i \qquad \text{(predictor)}$$

$$y_{i+1} = y_i + \frac{f_i + f(t_{i+1}, y_{i+1}^{0})}{2}\, \Delta t \qquad \text{(corrector)}$$

(4.) Midpoint Method

$$y_{i+1/2} = y_i + f_i \frac{\Delta t}{2}$$

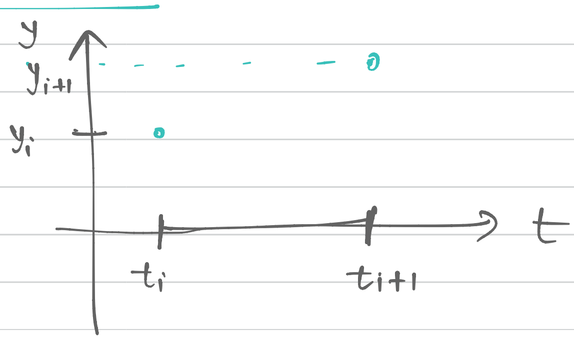
$$y_{i+1} = y_i + f(t_{i+1/2}, y_{i+1/2})\, \Delta t$$

Notation

$t_{i+1/2} = t_i + \Delta t/2$

$y_{i+1/2} = y(t_{i+1/2})$

- Error in forward Euler



$$y_{i+1} = y(t_{i+1})$$

$$= y(t_i) + \frac{dy}{dt}(t_i)\,(t_{i+1} - t_i)$$

$$+ \frac{1}{2!} \frac{d^2y}{dt^2}(t_i)\,(t_{i+1} - t_i)^2$$

$$+ \frac{1}{3!} \frac{d^3y}{dt^3}(t_i)\,(t_{i+1} - t_i)^3$$

$$+ \cdots + \frac{1}{n!} \frac{d^n y}{dt^n}(t_i)(t_{i+1} - t_i)^n + \cdots$$

Notation

$$y^{(k)}(t_i) = \frac{d^k y}{dt^k}(t_i)$$

$$\Rightarrow \quad y_{i+1} = y_i + \Delta t \, y^{(1)}(t_i) + \frac{\Delta t^2}{2!} y^{(2)}(t_i) + \cdots + \frac{\Delta t^n}{n!} y^{(n)}(t_i) + \cdots$$

$$= y_i + \Delta t \, y^{(1)}(t_i) + \frac{\Delta t^2}{2!} y^{(2)}(t_i) + \cdots + \frac{\Delta t^{(n-1)}}{(n-1)!} y^{(n-1)}(t_i) + \frac{\Delta t^n}{n!} y^{(n)}(z_n)$$

where $z_n \in [t_i, t_{i+1}]$ (mean field variable)

$$= y_i + \Delta t \, y^{(1)}(t_i) + \frac{\Delta t^2}{2!} y^{(2)}(t_i) + \cdots + \frac{\Delta t^{n-1}}{(n-1)!} y^{(n-1)}(t_i) + O(\Delta t^n)$$

$$y^{(1)}(t_i) = \frac{dy}{dt}(t_i) = f(t_i, y_i)$$

$$y^{(2)}(t_i) = \frac{d}{dt}\left(\frac{dy}{dt}(t_i)\right) = \frac{\partial}{\partial t} f(t_i, y_i) = f^{(1)}(t_i, y_i)$$

$$\rightarrow \boxed{y_{i+1} = y_i + \Delta t \, f_i} + \boxed{\frac{\Delta t^2}{2!} f^{(1)}(t_i, y_i) + \cdots + O(\Delta t^n)}$$

we ignore this $\longrightarrow O(\Delta t^2)$

So the local error (error in one step) is $O(\Delta t^2)$

Therefore the global error (error over all steps, $\frac{T}{\Delta t}$)

is $\frac{T}{\Delta t} O(\Delta t^2) = O(\Delta t)$

Error in Heun's method (Trapezoidal method)

$$\underbrace{\frac{dy}{dt} = f(t, y)}_{\downarrow}, \qquad y_i = y(t_i) \checkmark$$

$$y_{i+1} = y(t_{i+1}) \; ?$$

integrate it over $[t_i, t_{i+1}]$

$$\int_{t_i}^{t_{i+1}} \frac{dy}{dt} dt = \int_{t_i}^{t_{i+1}} f(t, y) dt$$

$$\Rightarrow y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} f(t, y) dt$$

$$\Rightarrow y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} f(t, y) dt .$$

We assume that, $f(t, y) = f(t)$

$$y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} f(t) dt \approx \frac{f(t_i) + f(t_{i+1})}{2} \Delta t$$

trapezoidal rule

$$y_{i+1} - y_i = \left(\frac{f_i + f_{i+1}}{2}\right) \Delta t$$

Recall that $$\int_a^b f(t)\,dt = \frac{f(b)+f(a)}{2}(b-a) - \frac{1}{12} f''(x)(b-a)^3$$

therefore $$\int_{t_i}^{t_{i+1}} f(t)\,dt = \frac{f_i+f_{i+1}}{2}\Delta t - \frac{1}{12} f''(x)\,\Delta t^3$$

$$\therefore\quad y_{i+1} - y_i = \frac{f_i+f_{i+1}}{2}\Delta t - \frac{1}{12} f''(x)\,\Delta t^3$$

$$\Rightarrow\quad \boxed{y_{i+1} = y_i + \frac{f_i+f_{i+1}}{2}\Delta t}\ \boxed{-\frac{1}{12} f''(x)\,\Delta t^3}$$

error $O(\Delta t^3)$

So local error is $O(\Delta t^3)$

$\therefore$ global error is $O(\Delta t^2)$

Error in backward Euler method

local error is $O(\Delta t^2)$

global error is $O(\Delta t)$.