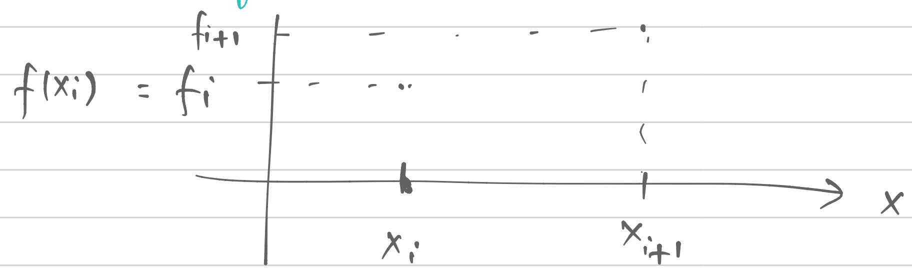
lecture 34

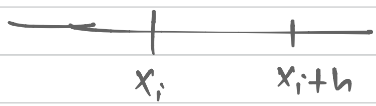
Approximation of differentials



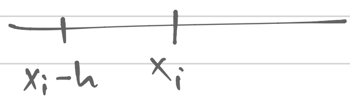
$$f'(x) = \frac{df(x)}{dx} = \dot{f}(x) = f^{(1)}(x)$$

to approximate $f'(x_i)$, $f'(x_{i+1})$

$$f'(x_i) = \lim_{h\to 0} \underbrace{\frac{f(x_i+h) - f(x_i)}{h}}_{\text{forward difference}}$$



$$= \lim_{h\to 0} \underbrace{\frac{f(x_i) - f(x_i-h)}{h}}_{\text{backward difference}}$$



"h is a positive number"

- $$\boxed{f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}}$$
- $$f'(x_{i+1}) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$
$$+ \frac{1}{2!} f''(x_i)(x_{i+1} - x_i)^2$$
$$+ \frac{1}{3!} f'''(x_i)(x_{i+1} - x_i)^3$$
$$+ \vdots$$

$$h = x_{i+1} - x_i \quad \Big| \Longrightarrow \quad f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + O(h^4)$$

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \left(\frac{f'''(x_i)h^3}{3!} + \cdots\right)$$

$$\left(\frac{f'''(x_i)h^3}{3!} + \frac{f^{iv}(x_i)}{4!}h^4 + \cdots\right)$$

$$O(h^3)$$

$$a = b_1 + \epsilon b_2 h + \epsilon^2 h^2 b_3 + \epsilon^3 h^3 b_4 + \cdots$$
$$= b_1 + O(\epsilon h)$$
$$= b_1 + \epsilon h b_2 + O(\epsilon^2 h^2)$$
$$= b_1 + \epsilon h b_2 + \epsilon^3 h^3 b_3 + O(\epsilon^4 h^4)$$

$$f(x_{i+1}) = f(x_i) + f'(x_i)\,h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + \frac{f^{iv}(x_i)}{4!}h^4 + \cdots$$

$$\Rightarrow f(x_{i+1}) = f(x_i) + O(h)$$

$$= f(x_i) + f'(x_i)\,h + O(h^2) \quad \text{— (1)}$$

$$= f(x_i) + f'(x_i)\,h + \frac{f''(x_i)}{2!}h^2 + O(h^3) \quad \text{— (2)}$$

$$= f(x_i) + f'(x_i)\,h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + O(h^4) \quad \text{— (3)}$$

from (1)

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{1}{h}O(h^2)$$

$$\Rightarrow \boxed{f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)}$$

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}$$

- $\frac{1}{h}O(h^n) = O(h^{n-1})$

$$O(h^2) = \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + \frac{f^{iv}(x_i)}{4!}h^4 + \cdots$$

$$\frac{1}{h}O(h^2) = \frac{f''(x_i)}{2!}h + \frac{f'''(x_i)}{3!}h^2 + \frac{f^{iv}(x_i)}{4!}h^3 + \cdots$$

$$= O(h)$$

- $(-1)\, O(h^n) = O(h^n)$

- if α is some fixed number
  $\alpha\, O(h^n) = O(h^n)$

- $h\, O(h^n) = O(h^{n+1})$

from (2)

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)}{2} h - O(h^3)\frac{1}{h}$$

$$\Rightarrow f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)}{2} h + O(h^2) \quad \text{— (4)}$$

Suppose we have three data points

$(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, $(x_{i+2}, f(x_{i+2}))$

s.t. $x_{i+1} - x_i = x_{i+2} - x_{i+1} = h$

then

$$\boxed{f''(x_i) \approx \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}}$$

from (4) $$f'(x_i) \approx \frac{f_{i+1} - f_i}{h} - \frac{1}{2}\left(\frac{f_{i+2} - 2f_{i+1} + f_i}{h^2}\right)h + O(h^2)$$

$$= \frac{1}{2h}\left(2f_{i+1} - 2f_i - f_{i+2} + 2f_{i+1} - f_i\right)$$

$$+ O(h^2)$$

$$\boxed{f'(x_i) \approx \frac{1}{2h}\left(-f_{i+2} + 4f_{i+1} - 3f_i\right) + O(h^2)}$$

$$f'(x_i) \approx \frac{1}{2h}\left(-f_{i+2} + 4f_{i+1} - 3f_i\right)$$