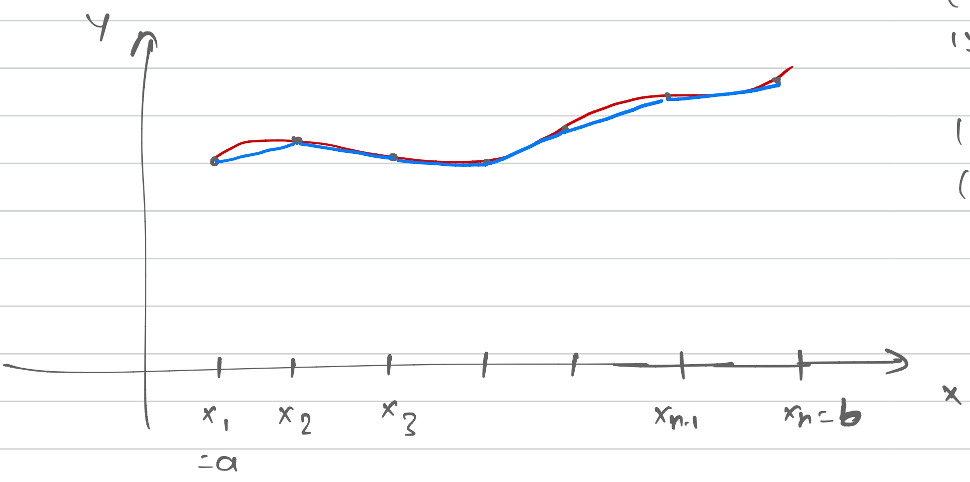
Lecture 30

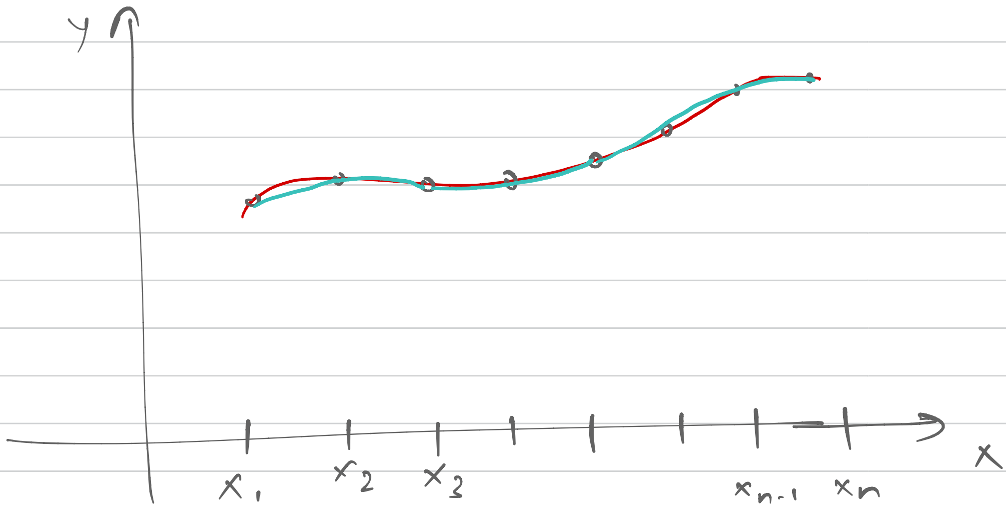
piecewise interpolation



$\left.\begin{matrix}(x_1, x_2) \\ (y_1, y_2)\end{matrix}\right\} \rightarrow$ line

$\left.\begin{matrix}(x_2, x_3) \\ (y_2, y_3)\end{matrix}\right\} \rightarrow$ line

⋮



$\left.\begin{matrix}(x_1, x_2, x_3) \\ (y_1, y_2, y_3)\end{matrix}\right\} \rightarrow$ parabola

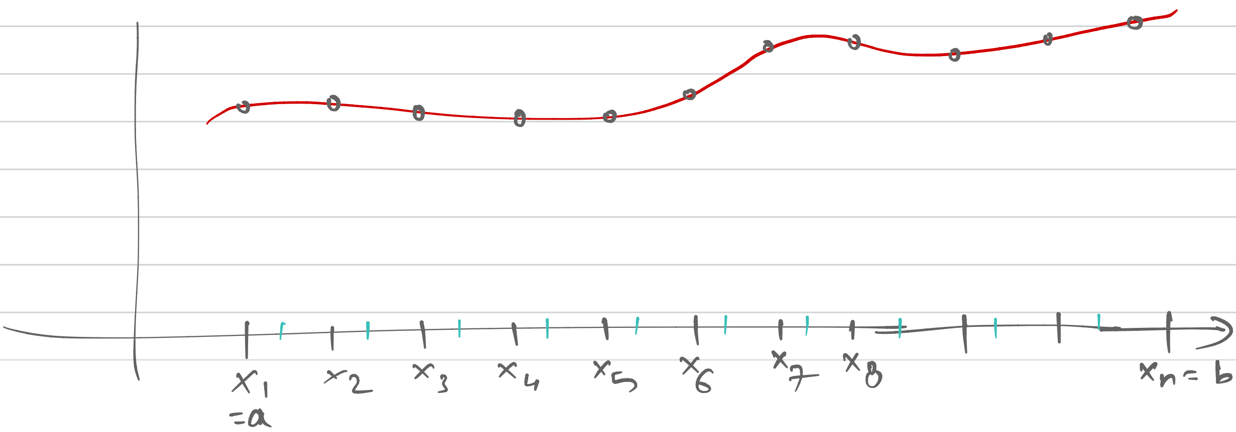
$\left.\begin{matrix}(x_3, x_4, x_5) \\ (y_3, y_4, y_5)\end{matrix}\right\} \rightarrow$ parabola

⋮

- Use idea of piecewise interpolation to approximate integration

$f : [a, b] \longrightarrow (-\infty, \infty)$

$$I[f] = \int_a^b f(x)\, dx,$$

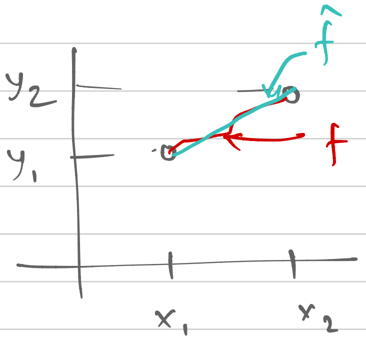


Linear interpolation

$$I[f] = \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

(x_1, y_1), (x_2, y_2) $\rightarrow$ fit a line $\hat{f}$

$$\int_{x_1}^{x_2} f(x)\,dx \approx \int_{x_1}^{x_2} \hat{f}(x)\,dx$$



$$= \int_{x_1}^{x_2} \left[y_1 \left(\frac{x - x_2}{x_1 - x_2} \right) + y_2 \left(\frac{x - x_1}{x_2 - x_1} \right) \right] dx$$

$$= \frac{(x_2 - x_1)}{2} \left[y_1 + y_2 \right]$$

Similarly,

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \int_{x_i}^{x_{i+1}} \hat{f}(x)\,dx = \frac{(x_{i+1} - x_i)}{2} \left[y_i + y_{i+1} \right]$$

$$\Rightarrow I[f] \approx \frac{(x_2 - x_1)}{2}[y_2 + y_1] + \frac{(x_3 - x_2)}{2}[y_3 + y_2] + \cdots + \frac{(x_n - x_{n-1})}{2}[y_n + y_{n-1}]$$

- if you assume

$x_2 - x_1 = h$

$x_3 - x_2 = h$

$\vdots$

$x_n - x_{n-1} = h$

then

$$I[f] \approx \frac{h}{2}\left[y_1 + y_n + 2\sum_{i=2}^{n-1} y_i \right]$$

Quadratic interpolation

$$I[f] = \int_{x_1}^{x_3} f(x)\,dx + \int_{x_3}^{x_5} f(x)\,dx + \int_{x_5}^{x_7} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

(x_1, y_1), (x_2, y_2), (x_3, y_3) $\rightarrow$ fit a parabola $\hat{f}$

$$\int_{x_1}^{x_3} f(x)\,dx \approx \int_{x_1}^{x_3} \hat{f}(x)\,dx$$

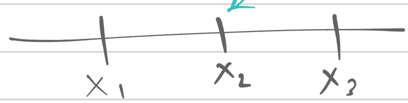
$$\hat{f}(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Lagrange interpolation

Assume that

$$x_2 = \frac{x_1 + x_3}{2} \Rightarrow (x_2 - x_1) = (x_3 - x_2) = \frac{(x_3 - x_1)}{2}$$

at mid point of $[x_1, x_3]$



$$\int_{x_1}^{x_3} f(x)\,dx \approx \frac{(x_3 - x_1)}{6}\left[y_1 + 4y_2 + y_3 \right]$$

If I denote $x_2 - x_1 = h$

i.e. $x_2 - x_1 = x_3 - x_2 = \frac{(x_3 - x_1)}{2} = h$

then

$$= \frac{h}{3}\left[y_1 + 4y_2 + y_3 \right]$$

Quadratic interpolation continued ...

$$\int_{x_i}^{x_{i+2}} f(x)\,dx \approx \frac{h}{3}\left[y_i + 4y_{i+1} + y_{i+2} \right]$$

where $h = x_{i+1} - x_i = x_{i+2} - x_{i+1}$

$$I[f] = \int_{x_1}^{x_3} f(x)\,dx + \int_{x_3}^{x_5} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$\approx \frac{h}{3}\left[y_1 + 4y_2 + y_3 \right] + \frac{h}{3}\left[y_3 + 4y_4 + y_5 \right]$$

$$+ \cdots + \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n \right]$$

$$= \boxed{\frac{h}{3}\left[y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n \right]}$$

Simpson's $\frac{1}{3}$ rd rule.

Cubic interpolation

$$I[f] = \int_{x_1}^{x_4} f(x)\,dx + \int_{x_4}^{x_7} f(x)\,dx + \cdots + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

(x_1, y_1)
(x_2, y_2)
(x_3, y_3)
(x_4, y_4) $\rightarrow$ fit a cubic function $\hat{f}$

$$\int_{x_1}^{x_4} f(x)\,dx \approx \int_{x_1}^{x_4} \hat{f}(x)\,dx = \frac{3}{8}h\left[y_1 + 3y_2 + 3y_3 + y_4 \right]$$

Simpson's $\frac{3}{8}$ th rule.

Errors due to numerical integration

- Trapezoidal rule (linear interpolation)

let say $y_i = f(x_i)$, $I[f] = \int_a^b f(x)\,dx$

$$I[f] \approx \underbrace{\frac{h}{2}\left[y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n \right]}_{\hat{I}[f]}$$

$$E_t = I[f] - \hat{I}[f]$$

$$= -\frac{(b-a)^3}{12n^3} \sum_{i=1}^{n-1} f''(z_i)$$

z_i is some point in interval $[x_i, x_{i+1}]$

$$f'(x) = \frac{df}{dx} \qquad f''(x) = \frac{d^2 f}{dx^2}$$

if $\bar{f}^{(k)} = \frac{1}{n} \sum_{i=1}^{n-1} f^{(k)}(z_i)$, $f^{(k)} = \frac{d^k f}{dx^k}$

$$E_t = -\frac{(b-a)^3}{12n^2}\bar{f}^{(2)} \to \bar{f}''$$

(f'' & $f^{(2)}$ same notation)

$f =$ const. E_t ? 0

$f =$ linear. E_t ? 0

$f =$ cubic. E_t ? some errors

Convergence rate as $n \to \infty$

$\frac{1}{n^2} \to 0$ higher rate compared to $\frac{1}{n}$

$\frac{1}{n} \to 0$

$\frac{1}{n^3} \to 0$ higher rate compared to $\frac{1}{n}$, $\frac{1}{n^2}$

- Simpson's $\frac{1}{3}$ rule (Quadratic interpolation)

$$E_t = I[f] - \hat{I}[f]$$

$$= -\frac{(b-a)^5}{180\, n^4} \bar{f}^{(4)}$$

$f =$ const
$f =$ linear
$f =$ quadratic
$f =$ cubic

$\longrightarrow E_t = 0$ /

$n \to 0$

rate of convergence

$\frac{1}{n^4} \longrightarrow 0$

$\left(\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3} \right)$

Trapezoidal rate of convergence