Lecture 29

Lagrange's interpolation method

$(x_1, y_1), \ldots, (x_{n+1}, y_{n+1})$

$$\hat{y} = \hat{f}(x) = Z(x) \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{bmatrix} \rightarrow n^{th} \text{ order polynomial}$$

$$Z(x) = [L_1(x), L_2(x), \ldots, L_{n+1}(x)]$$

$$L_i(x) = \frac{\sum_{\substack{j=1 \\ j \neq i}}^{n+1} (x - x_j)}{\sum_{\substack{j=1 \\ j \neq i}}^{n+1} (x_i - x_j)}$$

$$a_1 = y_1, \quad a_2 = y_2, \ldots, \quad a_{n+1} = y_{n+1}$$

$$J a = b, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{bmatrix}, \quad a -$$

$$J = \begin{bmatrix} \text{---} & Z(x_1) & \text{---} \\ \text{---} & Z(x_2) & \text{---} \\ & & \\ \text{---} & Z(x_{n+1}) & \text{---} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & & 0 \\ & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

• Linear function $\quad (x_1, y_1), \quad (x_2, y_2)$

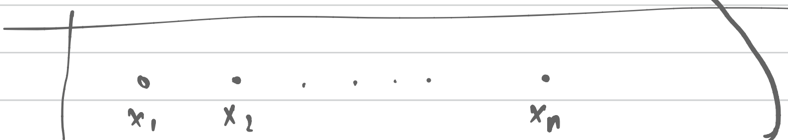
$$\hat{y} = \hat{f}(x) = a_1 L_1(x) + a_2 L_2(x) \longrightarrow \text{Lagrange's interpolation}$$

$$\tilde{y} = \tilde{f}(x) = b_1 + b_2 (x - x_1) \longrightarrow \text{Newton's interpolation}$$

Property 1 :
$$\hat{y}(x_1) = a_1 = y_1, \quad \hat{y}(x_2) = a_2 = y_2$$

$$\tilde{y}(x_1) = b_1 = y_1, \quad \tilde{y}(x_2) = b_1 + b_2(x_2 - x_1) = y_2$$

$$k \frac{d^2 T}{dx^2} = q$$



$$T_i = T(x_i)$$

$$\hat{T}(x) = \sum_{i=1}^{n} a_i L_i(x)$$ (Approximating temperature fied T using $\hat{T}$)

$$J a = b, \qquad a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \qquad b = \begin{bmatrix} q(x_1) + T_0 \\ \vdots \\ q(x_n) + T_L \end{bmatrix}$$

$$J = \begin{bmatrix} \; - \; \end{bmatrix}$$

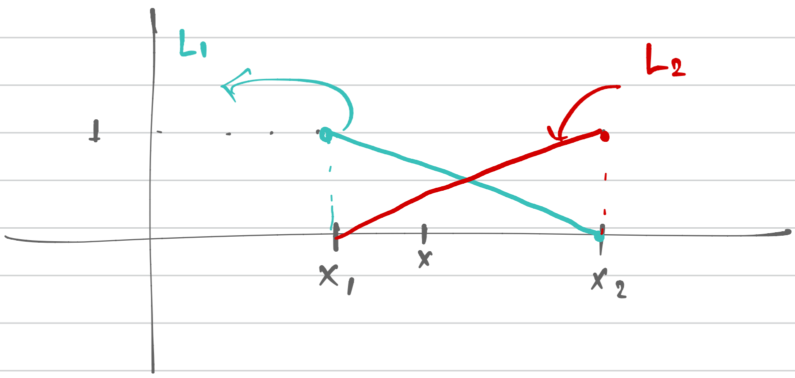
Because of Lagrange interpolation property, coefficients $a_1, a_2, \ldots, a_n$ are also the values of temperature at $x_1, x_2, \ldots, x_n$!!

Property 2

Linear example

$$\hat{y} = y_1 L_1(x) + y_2 L_2(x)$$

$$= y_1 \frac{x - x_2}{x_1 - x_2} + y_2 \frac{x - x_1}{x_2 - x_1}$$



$L_1(x_1) = 1$

$L_1(x_2) = 0$

$L_2(x_1) = 0$

$L_2(x_2) = 1$

$$L_1(x) + L_2(x) = \frac{x - x_2}{x_1 - x_2} + \frac{x - x_1}{x_2 - x_1}$$

$$= \frac{1}{(x_2 - x_1)} \left[x - x_1 - x + x_2 \right]$$

$$= \frac{x_2 - x_1}{x_2 - x_1} = 1$$

$$\Rightarrow \underbrace{L_1(x) + L_2(x) = 1}_{\text{Crucial to exactly interpolate any constant function}} \quad \text{for any } x \in [x_1, x_2]$$

constant function $\rightarrow$ $y = y(x) = C$, $y_1 = C$, $y_2 = C$

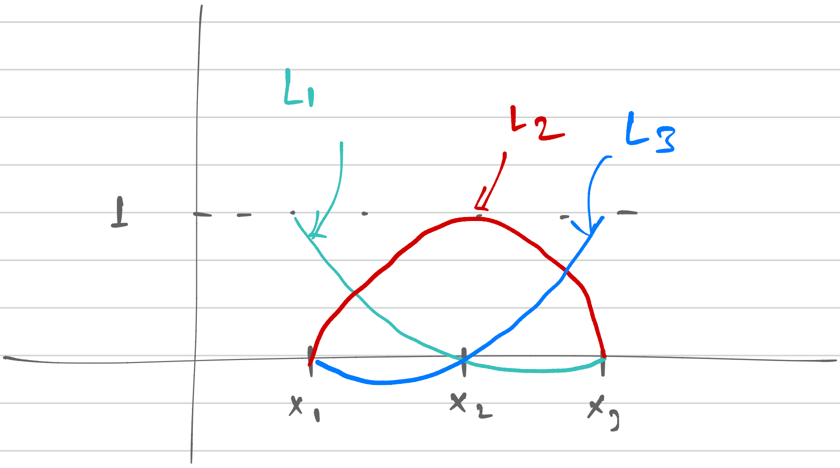
$$\hat{y} = y_1 L_1(x) + y_2 L_2(x) = C(L_1(x) + L_2(x)) = C \checkmark$$

- Quadratic example

$$\hat{y} = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

$$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}, \quad L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$



$L_1(x_1) = 1$

$L_1(x_2) = 0$

$L_1(x_3) = 0$

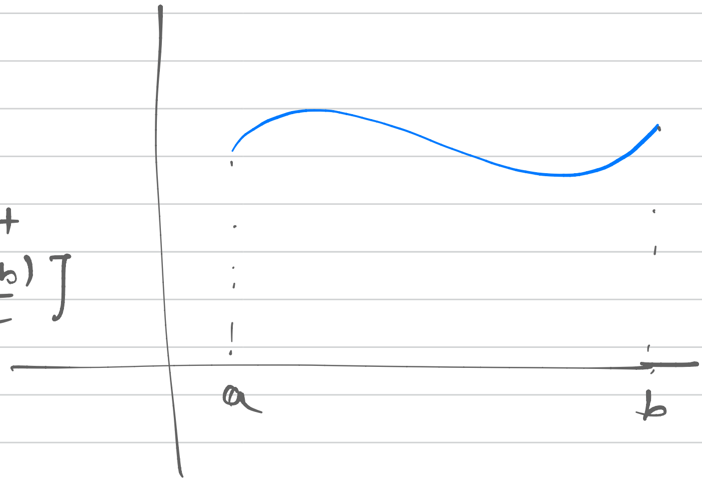
- Numerical approximation of integrals

$$f : [a,b] \to (-\infty, \infty)$$

$$I[f] = \int_a^b f(x)\,dx$$

$$\simeq h\left[\frac{f(a)}{2} + f(a+h) + f(a+2h) + \dots + f(b-h) + \frac{f(b)}{2}\right]$$

"Riemann"



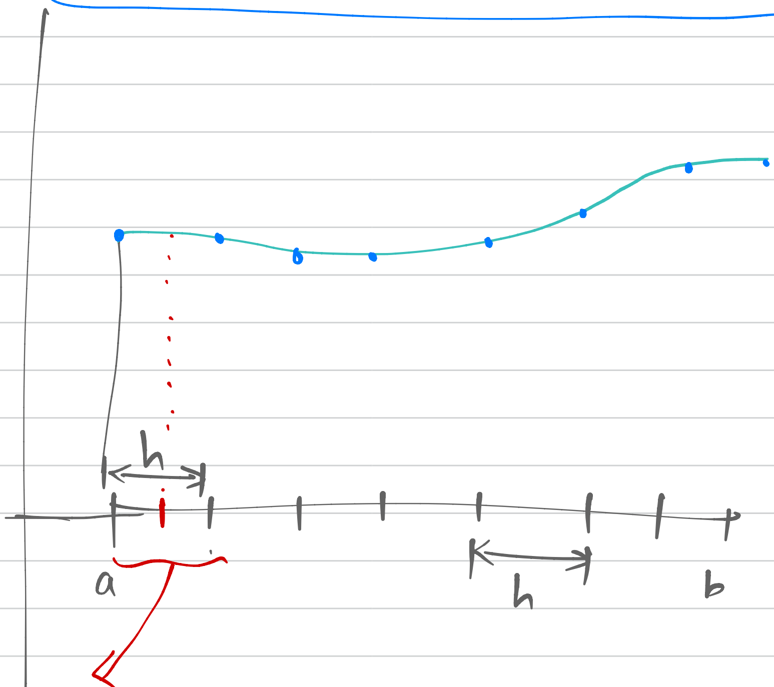
Definition of integral

$$I[f] = \lim_{h\to 0}\left[h\left(\frac{f(a)+f(a+h)}{2}\right) + h\left(\frac{f(a+h)+f(a+2h)}{2}\right) + \vdots + h\left(\frac{f(b-h)+f(b)}{2}\right)\right]$$

use the definition but without " $\lim_{h\to 0}$ "

$$I[f] = \int_a^b f(x)\,dx$$

$$\simeq \frac{h}{2}\Big[f(a) + 2f(a+h) + 2f(a+2h) + \dots + 2f(a+(n-1)h) + f(b)\Big]$$



Note also (we can break integration into smaller integrals)

$$I[f] = \int_a^b f(x)\,dx$$

$$= \int_a^{a+h} f(x)\,dx + \int_{a+h}^{a+2h} f(x)\,dx + \dots + \int_{a+(n-1)h}^{b} f(x)\,dx$$

$$\int_a^{a+h} f(x)\,dx \simeq h\left(\frac{f(a)+f(a+h)}{2}\right)$$

$$\int_{a+h}^{a+2h} f(x)\,dx \simeq h\left(\frac{f(a+h)+f(a+2h)}{2}\right)$$

for general method of integration

$$I[f] = \int_a^b f(x)\,dx$$

$(a = x_1, x_2, \ldots, x_{n-1}, x_n = b)$ ← discrete points

$(f(x_1), f(x_2), \ldots, f(x_{n-1}), f(x_n))$ ← discrete values of function

$$I[f] = \int_a^b f(x)\,dx = \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$\int_{x_1}^{x_2} f(x)\,dx$, $\boxed{(x_1, f(x_1)), (x_2, f(x_2))}$

↓

linear interpolation

$$\hat{f}(x) = f(x_1)\frac{x - x_2}{x_1 - x_2} + f(x_2)\frac{x - x_1}{x_2 - x_1}$$

use linear interpolation to approximate

$$\approx \int_{x_1}^{x_2} \hat{f}(x)\,dx = f(x_1)\int_{x_1}^{x_2} \frac{(x - x_2)}{(x_1 - x_2)}\,dx + f(x_2)\int_{x_1}^{x_2} \frac{(x - x_1)}{(x_2 - x_1)}\,dx$$

$$= (x_2 - x_1)\left[\frac{f(x_1) + f(x_2)}{2}\right]$$

$$= \frac{1}{(x_2 - x_1)}\left[-f(x_1)\left[\int_{x_1}^{x_2} x\,dx - \int_{x_1}^{x_2} x_2\,dx\right] + f(x_2)\left[\int_{x_1}^{x_2} x\,dx - \int_{x_1}^{x_2} x_1\,dx\right]\right]$$

$$+\frac{x_1^2}{2}+\frac{x_2^2}{2} - x_1 x_2$$

$$= \frac{1}{(x_2 - x_1)} \Big[-f(x_1)\Big(\overbrace{\frac{x_2^2}{2} - \frac{x_1^2}{2} - x_2^2 + x_2 x_1}\Big) + f(x_2)\Big(\frac{x_2^2 - x_1^2}{2} - x_1 x_2 + x_1^2\Big)\Big]$$

$$= \frac{1}{(x_2 - x_1)} \Big[\frac{f(x_1)}{2}\left(-2x_2 x_1 + x_1^2 + x_2^2\right) + \frac{f(x_2)}{2}\left(x_2^2 + x_1^2 - 2x_1 x_2\right)\Big]$$

segment $[x_1, x_2]$

$$= \frac{1}{(x_2 - x_1)} \left[\frac{f(x_1)}{2} + \frac{f(x_2)}{2}\right](x_1 - x_2)^2$$

$$= (x_2 - x_1)\left[\frac{f(x_1)}{2} + \frac{f(x_2)}{2}\right]$$

$$= h\left[\frac{f(a)}{2} + \frac{f(a+h)}{2}\right]$$

$x_1 = a$

$x_2 = a + h$

$x_3 = a + 2h$

$\vdots$

segment $[x_2, x_3]$

$$\int_{x_2}^{x_3} \hat{f}(x)\,dx = h\left[\frac{f(a+h) + f(a+2h)}{2}\right]$$

for segment (x_i, x_{i+1})

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \quad , \quad \underbrace{(x_i, f(x_i)), \ (x_{i+1}, f(x_{i+1}))}$$

$$\hat{f}(x) = f(x_i) \frac{x - x_{i+1}}{x_i - x_{i+1}} + f(x_{i+1}) \frac{x - x_i}{x_{i+1} - x_i}$$

$$\approx \int_{x_i}^{x_{i+1}} \hat{f}(x)\,dx$$

$$= (x_{i+1} - x_i) \left[\frac{f(x_i) + f(x_{i+1})}{2} \right]$$

Combining

$$I[f] \approx \boxed{\frac{1}{2} \Big[(x_2 - x_1)\big(f(x_2) + f(x_1)\big) + (x_3 - x_2)\big(f(x_3) + f(x_2)\big) + \dots + (x_{n-1} - x_n)\big(f(x_{n-1}) + f(x_n)\big) \Big]}$$

for general discretization

$$\underbrace{x_1 = a, \ x_2 = a+h, \ \dots, \ x_{n-1} = b-h, \ x_n = b}$$

$$\Downarrow$$

$$= \boxed{h \left[\frac{f(a)}{2} + f(a+h) + f(a+2h) + \dots + f(b-h) + \frac{f(b)}{2} \right]}$$

for uniform discretization.