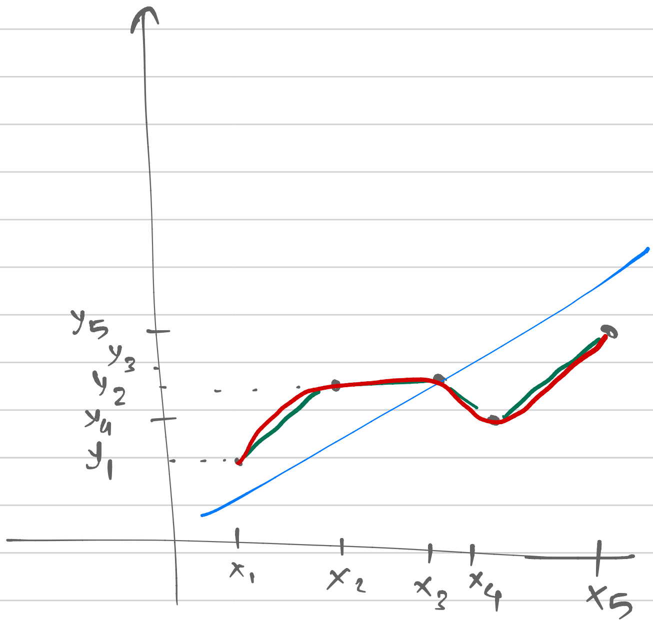
Lecture 27

Interpolation

$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$

find a curve $f = f(x)$ such that



$f(x_1) = y_1$

$f(x_2) = y_2$

$\vdots$

$f(x_n) = y_n$

<u>Idea</u> take f function as polynomial in x.

$z(x)\, a = f(x) = a_1 + a_2 x + a_3 x^2 + \dots + a_n x^{n-1}$

$z = [1, x, x^2, \dots, x^{n-1}]$

Find unknowns $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ s.t

(1) $f(x_1) = y_1 = a_1 + a_2 x_1 + a_3 x_1^2 + \dots + a_n (x_1)^{n-1}$

(2) $f(x_2) = y_2 = a_1 + a_2 x_2 + a_3 x_2^2 + \dots + a_n (x_2)^{n-1}$

$\vdots$

(n) $f(x_n) = y_n = a_1 + a_2 x_n + a_3 x_n^2 + \dots + a_n (x_n)^{n-1}$

$$J a = b$$

$$J = \begin{bmatrix} — & z(x_1) & — \\ — & z(x_2) & — \\ & \vdots & \\ — & z(x_n) & — \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

$x_1 < x_2 < \cdots x_3 < x_n$

$=1 \quad =2 \quad =5 \quad =10$

$n = 4$

$$J = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 5 & 25 & 125 \\ 1 & 10 & 100 & 1000 \end{bmatrix}$$

$Ax = b$

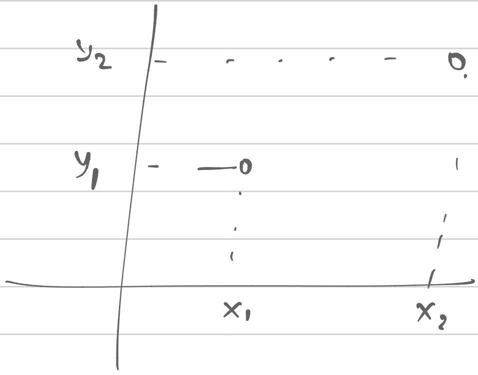
error in x $\leq$ cond$[A]$ * error in b

• Newton's interpolation method

$$f(x) = a_1 + a_2 (x - x_1)$$

" direct method

$$f(x) = a_1 + a_2 x \text{ "}$$



• $f(x_1) = y_1 = a_1 + a_2 (x_1 - x_1) = a_1 \Rightarrow a_1 = y_1$

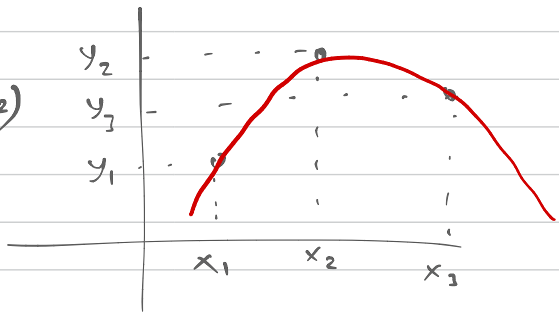
• $f(x_2) = y_2 = a_1 + a_2 (x_2 - x_1) = y_1 + a_2 (x_2 - x_1)$

$$\Rightarrow a_2 = \frac{y_2 - y_1}{x_2 - x_1}$$

Example of quadratic polynomial

$$f(x) = a_1 + a_2 (x - x_1) + a_3 (x - x_1)(x - x_2)$$

" $f(x) = a_1 + a_2 x + a_3 x^2$ "



• $f(x_1) = y_1 \Rightarrow y_1 = a_1$

• $f(x_2) = y_2 \Rightarrow y_2 = a_1 + a_2 (x_2 - x_1) \Rightarrow a_2 = \dfrac{y_2 - y_1}{x_2 - x_1}$

• $f(x_3) = y_3 \Rightarrow y_3 = a_1 + a_2 (x_3 - x_1) + a_3 (x_3 - x_1)(x_3 - x_2)$

$$\Rightarrow y_3 - y_1 - \frac{(y_2 - y_1)}{(x_2 - x_1)} (x_3 - x_1) = a_3 (x_3 - x_1)(x_3 - x_2)$$

$$\Rightarrow \boxed{a_3 = \frac{\dfrac{(y_3 - y_1)}{(x_3 - x_1)} - \dfrac{(y_2 - y_1)}{(x_2 - x_1)}}{x_3 - x_2}}$$

$$a_3 (x_3 - x_1)(x_3 - x_2) = y_3 - y_1 - \frac{(y_2 - y_1)}{(x_2 - x_1)} (x_3 - x_1)$$

$$= y_3 - y_2 + (y_2 - y_1) - \frac{(y_2 - y_1)}{(x_2 - x_1)} (x_3 - x_1)$$

$$= (y_3 - y_2) - (y_2 - y_1)\left[\frac{x_3 - x_1}{x_2 - x_1} - 1 \right]$$

$$\Rightarrow a_3 (x_3 - x_1)(x_3 - x_2) = (y_3 - y_2) - (y_2 - y_1)\left[\frac{x_3 - x_2}{x_2 - x_1} \right]$$

$$\Rightarrow a_3 = \frac{(y_3 - y_2)}{(x_3 - x_1)(x_3 - x_2)} - \frac{(y_2 - y_1)}{(x_3 - x_1)\cancel{(x_3 - x_2)}} \frac{\cancel{(x_3 - x_2)}}{(x_2 - x_1)}$$

$$= \frac{y_3 - y_2}{(x_3 - x_1)(x_3 - x_2)} - \frac{y_2 - y_1}{(x_3 - x_1)(x_2 - x_1)}$$

$$\boxed{a_3 = \frac{\dfrac{y_3 - y_2}{x_3 - x_2} - \dfrac{y_2 - y_1}{x_2 - x_1}}{x_3 - x_1}} = y[3, 2, 1]$$

$$f(x) = a_1 + a_2 (x - x_1) + a_3 (x - x_1)(x - x_2)$$

$$= \underbrace{[1, \; x - x_1, \; (x - x_1)(x - x_2)]}_{z(x)} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{a}$$

- $f(x_1) = y_1, \quad f(x_2) = y_2, \quad f(x_3) = y_3$

$$Ja = b, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$J = \begin{bmatrix} \text{---} & Z(x_1) & \text{---} \\ \text{---} & Z(x_2) & \text{---} \\ \text{---} & Z(x_3) & \text{---} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & x_2 - x_1 & 0 \\ 1 & x_3 - x_1 & (x_3 - x_1)(x_3 - x_2) \end{bmatrix}$$

- finite divided differences

$y_1, y_2, y_3, \ldots, y_n$

$x_1, x_2, x_3, \ldots, x_n$

- $y[i] = y_i$

- $y[j,i] = \dfrac{y_j - y_i}{x_j - x_i} \quad \rightarrow \quad y[2,1] = \dfrac{y_2 - y_1}{x_2 - x_1}$

- $y[k,j,i] = \dfrac{y[k,j] - y[j,i]}{x_k - x_i} \quad \rightarrow \quad y[3,2,1] = \dfrac{y[3,2] - y[2,1]}{x_3 - x_1}$

$$= \dfrac{\dfrac{y_3 - y_2}{x_3 - x_2} - \dfrac{y_2 - y_1}{x_2 - x_1}}{x_3 - x_1}$$

- $y[m,k,j,i] = \dfrac{y[m,k,j] - y[k,j,i]}{x_m - x_i}$

$\vdots$

- $y[n, n-1, n-2, \ldots, 3, 2, 1] = \dfrac{y[n, n-1, n-2, \ldots, 3, 2] - y[n-1, n-2, \ldots, 2, 1]}{x_n - x_1}$

- line

$$a_1 = y[1] = y_1$$

$$a_2 = y[2,1] = \frac{y_2 - y_1}{x_2 - x_1}$$

- quadratic

$$a_1 = y[1] = y_1$$

$$a_2 = y[2,1] = \frac{y_2 - y_1}{x_2 - x_1}$$

$$a_3 = y[3,2,1] = \frac{y[3,2] - y[2,1]}{x_3 - x_1}$$

- $(n-1)^{th}$ order polynomial $\longrightarrow (x_1, y_1), \ldots, (x_n, y_n)$

$$f(x) = a_1 + a_2(x-x_1) + a_3(x-x_1)(x-x_2) + \cdots + a_n(x-x_1)(x-x_2)\cdots(x-x_{n-1})$$

$a_1 = y[1] = y_1$

$a_2 = y[2,1]$

$a_3 = y[3,2,1]$

$a_4 = y[4,3,2,1]$

$\vdots$

$$a_n = y[n, n-1, n-2, \ldots, 3, 2, 1]$$

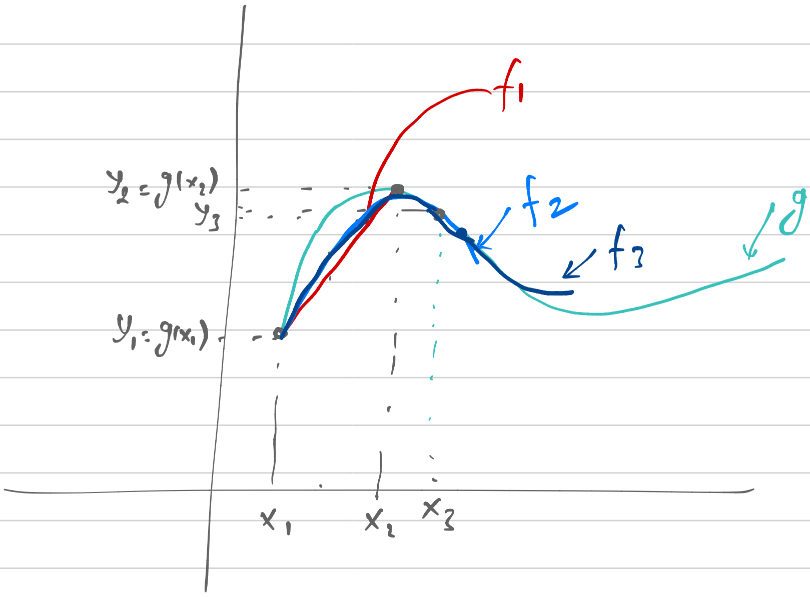
<u>Suppose</u> $\quad y = g(x)$

$$(x_1, y_1), (x_2, y_2) \ldots, (x_n, y_n)$$

$$(x_1, g(x_1)), (x_2, g(x_2)), \ldots, (x_n, g(x_n))$$

$(x_1, y_1), (x_2, y_2)$

$$f_1(x) = a_1 + a_2(x - x_1)$$



$(x_1, y_1), (x_2, y_2), (x_3, y_3)$

$$f_2(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2)$$

$$= f_1(x) + a_3(x - x_1)(x - x_2)$$

$$a_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{g(x_2) - g(x_1)}{x_2 - x_1} \approx \frac{dg}{dx}(x_1)$$

$$a_3 = \frac{\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1}}{x_3 - x_1}$$

$$\approx \frac{\frac{dg}{dx}(x_3) - \frac{dg}{dx}(x_1)}{x_3 - x_1} \approx \frac{d^2 g}{dx^2}(x_1)$$

Lecture 28 · Errors in interpolation

let data $y_i = f(x_i)$, "some function f"

let $\hat{y}_i = \hat{f}(x_i)$ interpolation function

Consider $(n+1)$ data points $(x_1, y_1), \ldots, (x_{n+1}, y_{n+1})$

then $\hat{f}$ would be n^{th} order polynomial

Error function $e(x) = f(x) - \hat{f}(x)$

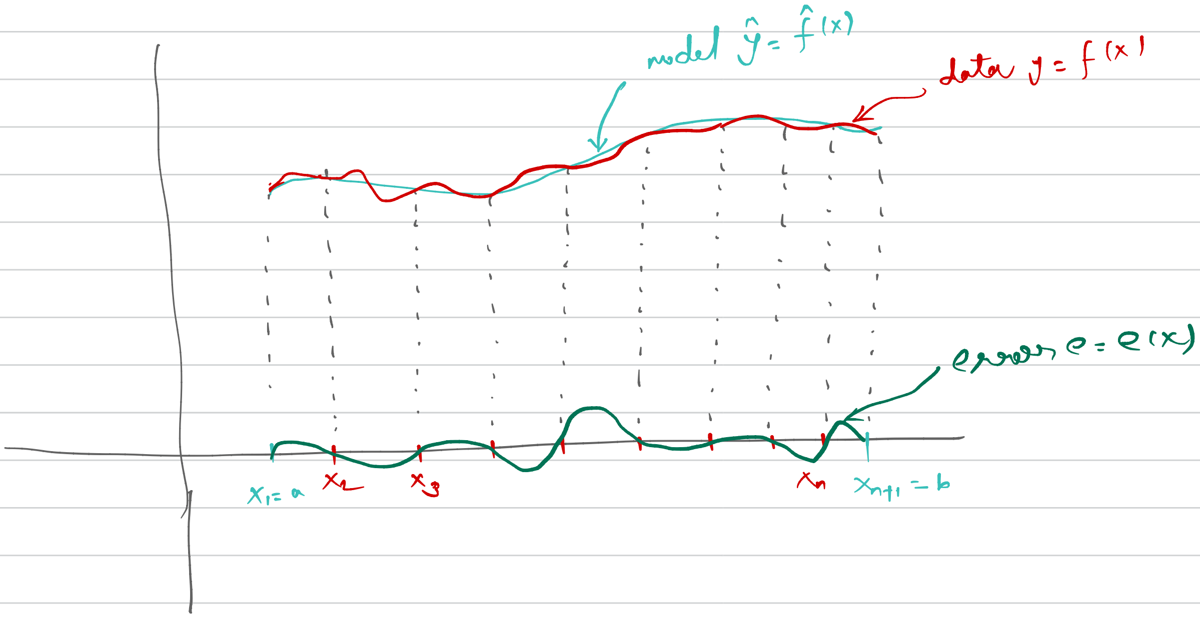
Particularly $e(x_1) = f(x_1) - \hat{f}(x_1) = 0$

$e(x_2) = f(x_2) - \hat{f}(x_2) = 0$

$\vdots$

$e(x_{n+1}) = f(x_{n+1}) - \hat{f}(x_{n+1}) = 0$

I.e. $e = e(x)$ must have $n+1$ roots



find H s.t

$$e(x) \approx H\,(x - x_1)(x - x_2) \cdot\cdot (x - x_n)$$

$$\Rightarrow f(x) - \hat{f}(x) = H(x - x_1)(x - x_2) \ldots (x - x_n)$$

$$\Rightarrow \frac{d^{n+1}}{dx^{n+1}} f(x) - \cancel{\frac{d^{n+1} \hat{f}(x)}{dx^{n+1}}}^{\,0} = H(n+1)!$$

$$\boxed{H = \frac{1}{(n+1)!} \frac{d^{n+1}}{dx^{n+1}} f(x)}$$

$$= H\left[x^{n+1} + \alpha_1 x^n + \alpha_2 x^{n-1} + \dots + \alpha_n\right]$$

$(n+1)^{th}$ derivative

$$\frac{d x^{n+1}}{dx} = (n+1)x^n$$

$$\frac{d^2 x^{n+1}}{dx^2} = (n+1)n x^{n-1}$$

$$\vdots$$

$$\frac{d^{n+1} x^{n+1}}{dx^{n+1}} = (n+1)n(n-1)\dots 3 \cdot 2 \cdot 1$$

$$= (n+1)!$$

$$e \approx \frac{1}{(n+1)!}\left[\frac{d^{n+1}}{dx^{n+1}} f(x)\right](x-x_1)(x-x_2) \dots (x-x_{n+1})$$

there is a number M s.t.

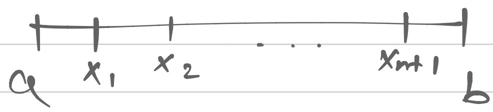
$$\left|\frac{d^{n+1}}{dx^{n+1}} f(x)\right| \le M \quad \text{for any } x \in [a,b]$$

assume that

$$x_1, x_2, \dots, x_{n+1} \in [a,b]$$

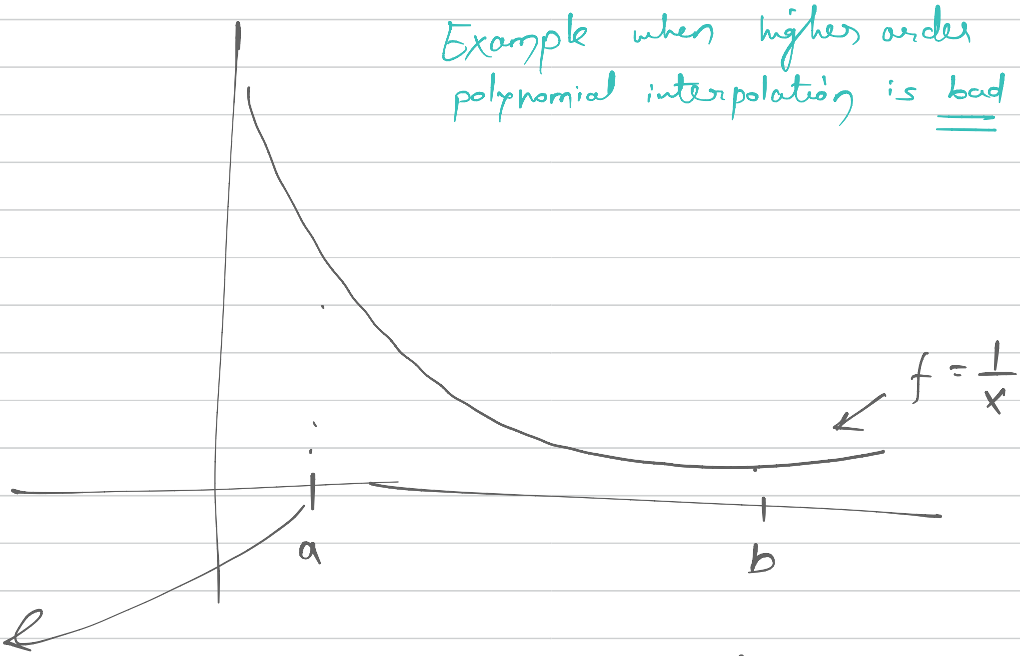
$$|e| \le \frac{M}{(n+1)!} \left|(x-x_1)(x-x_2)\dots(x-x_{n+1})\right|$$

$$|x - x_i| \le |b-a|$$



$$\boxed{|e| \le \frac{M}{(n+1)!} |b-a|^{n+1}}$$

Example when higher order polynomial interpolation is bad



$$\left| \frac{d^{n+1} f}{dx^{n+1}} \right| = \frac{|(-1)(-2)\cdots(-(n+1))|}{x^{n+2}}$$

$$\frac{d}{dx} \frac{1}{x} = \frac{-1}{x^2}$$

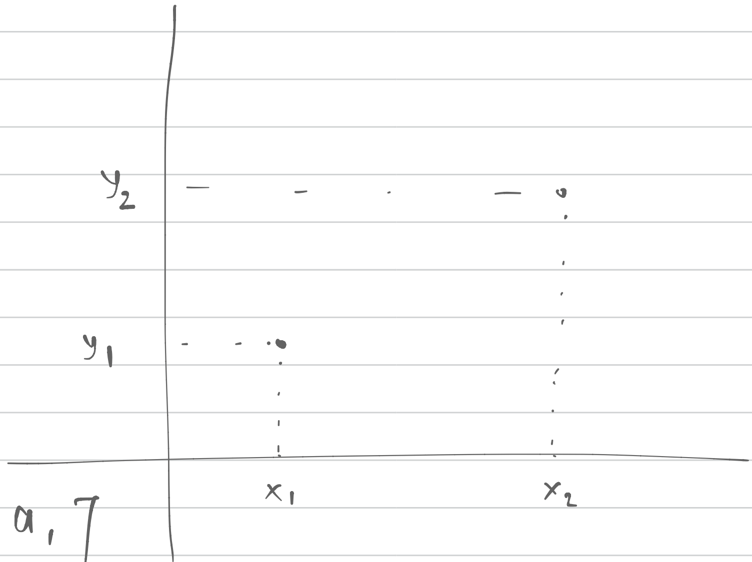
$$\frac{d^2}{dx^2} \frac{1}{x} = \frac{(-1)(-2)}{x^3}$$

$$= \frac{(n+1)!\, |(-1)^{n+1}|}{x^{n+2}}$$

$$= \frac{(n+1)!}{x^{n+2}} \leq \frac{(n+1)!}{a^{n+2}}$$

• Lagrange's interpolation method

(x_1, y_1) , (x_2, y_2) two data points



$$\hat{y} = \hat{f}(x) = [1, \; x - x_1] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$a_1 = y_1$$

$$a_2 = \frac{y_2 - y_1}{x_2 - x_1}$$

Newton's interpolation

$$\boxed{\hat{y} = \hat{f}(x) = y_1 + \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)}$$

$$\tilde{y} = \tilde{f}(x) = b_1 \underbrace{\left[\frac{x - x_2}{x_1 - x_2}\right]}_{L_1(x)} + b_2 \underbrace{\left[\frac{x - x_1}{x_2 - x_1}\right]}_{L_2(x)}$$

Lagrange interpolation

$$= \underbrace{[L_1(x), \; L_2(x)]}_{Z(x)} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\tilde{y}(x_1) = y_1 \Rightarrow L_1(x_1)\, b_1 + L_2(x_1)\, b_2 = y_1$$

$$\Rightarrow \left(\frac{x_1 - x_2}{x_1 - x_2}\right)^{1} b_1 + \left(\frac{x_1 - x_1}{x_2 - x_1}\right)^{0} b_2 = y_1$$

$$\Rightarrow \boxed{b_1 = y_1}$$

$$\tilde{y}(x_2) = y_2 \Rightarrow L_1(x_2)\, b_1 + L_2(x_2)\, b_2 = y_2$$

$$\Rightarrow \left(\frac{x_2 - x_2}{x_1 - x_2}\right)^{0} b_1 + \left(\frac{x_2 - x_1}{x_2 - x_1}\right)^{1} b_2 = y_2$$

$$\Rightarrow \boxed{b_2 = y_2}$$

Quadratic function $\quad (x_1, y_1),\ (x_2, y_2),\ (x_3, y_3)$

$$\tilde{y} = \tilde{f}(x) = Z(x) \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where

$$Z(x) = [\, L_1(x),\ L_2(x),\ L_2(x) \,]$$

$$L_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

$$L_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$L_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

- $\tilde{y}(x_1) = y_1 = L_1(x_1)^{1} b_1 + L_2(x_1)^{0} b_2 + L_3(x_1)^{0} b_3$

$$\Rightarrow \boxed{y_1 = b_1}$$

- $\tilde{y}(x_2) = y_2 = L_1(x_2)^{0} b_1 + L_2(x_2)^{1} b_2 + L_3(x_2)^{0} b_3$

$$- \boxed{y_2 = b_2}$$

- $\tilde{y}(x_3) = y_3$

$\Rightarrow \boxed{y_3 = b_3}$