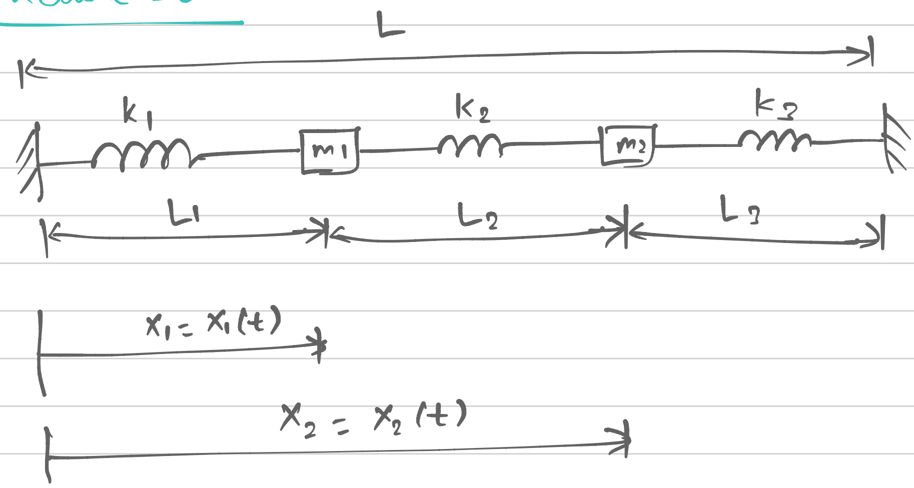
Lecture 20



$$L = L_1 + L_2 + L_3$$

IC
$$\begin{cases} x_1(0) = L_1 & \dot{x}_1(0) = v_0 \\ x_2(0) = L_1 + L_2 & \dot{x}_2(0) = w_0 \end{cases}$$

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 (x_1 - L_1) + k_2 (x_2 - x_1 - L_2)$$

$$m_2 \frac{d^2 x_2}{dt^2} = k_3 (L_1 + L_2 - x_2) - k_2 (x_2 - x_1 - L_2)$$

Change of variable

$y_1 = x_1 - L_1 \longrightarrow$ displacement of mass m_1

$y_2 = x_2 - (L_1 + L_2) \longrightarrow$ —//— m_2

IC
$$\begin{cases} y_1(0) = 0 & \frac{dy_1}{dt}(0) = \dot{x}_1(0) = v_0 \\ y_2(0) = 0 & \frac{dy_2}{dt}(0) = \dot{x}_2(0) = w_0 \end{cases}$$

$$\boxed{\begin{aligned} m_1 \frac{d^2 y_1}{dt^2} &= -k_1 y_1 + k_2 (y_2 - y_1) \\ m_2 \frac{d^2 y_2}{dt^2} &= -k_3 y_2 - k_2 (y_2 - y_1) \end{aligned}}$$

$$\boxed{m \frac{d^2u}{dt^2} = -a\,u}$$

$u(0) = 0$

$\dot{u}(0) = V_0$

$u(t) = \alpha \sin(\beta t)$

$$\frac{du}{dt} = \alpha\beta \cos(\beta t)$$

$$\boxed{\begin{aligned}\frac{d^2u}{dt^2} &= -\alpha\beta^2 \sin(\beta t) \\ &= -\beta^2 u\end{aligned}}$$

$$\beta^2 = \frac{a}{m} \Rightarrow \boxed{\beta = \sqrt{a/m}}$$

$u(0) = \alpha \sin(0) = 0$ ✓

$\dot{u}(0) = \alpha\beta\cos(0) = \alpha\beta = V_0 \Rightarrow \alpha\beta = V_0$

$$\Rightarrow \boxed{\alpha = V_0\sqrt{\frac{m}{a}}}$$

Vector notation

$$u = u(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \Rightarrow \frac{d}{dt}u(t) = \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{(k_1+k_2)}{m_1} & k_2/m_1 \\ \frac{k_2}{m_2} & -\frac{(k_3+k_2)}{m_2} \end{bmatrix} \qquad \frac{d^2u}{dt^2} = \begin{bmatrix} \frac{d^2y_1}{dt^2} \\ \frac{d^2y_2}{dt^2} \end{bmatrix}$$

$$\boxed{\frac{d^2u}{dt^2} = A\,u}$$

x_1, x_2, w are numbers and are constant

$$u(t) = \sin(wt)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{du}{dt} = w\cos(wt)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boxed{\frac{d^2u}{dt^2} = -w^2 \sin(wt)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -w^2 u}$$

I want x_1, x_2, w to be such that

$$Au = -w^2 u$$

$$\Rightarrow A\left(\sin(wt)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = -w^2 \sin(wt)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \cancel{\sin(wt)}\, A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cancel{\sin(wt)}\left(-w^2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$A(\alpha u) = \alpha(Au)$

$$\Rightarrow \boxed{A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -w^2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

+

$$\boxed{\lambda = -w^2}$$

$a_{11} = -\frac{(k_1+k_2)}{m_1}$

$a_{12} = k_2/m_1$

$a_{21} = k_2/m_2$

$a_{22} = -\frac{(k_2+k_3)}{m_2}$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow (A - \lambda I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$x_1 = 0, \quad x_2 = 0$$

If B is a matrix such that there is a vector $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ with property that

$$BX = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$ and not all elements of X are zero

Then B must be singular, i.e.,

$$\det(B) = 0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow$ from $(A - \lambda I)X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

then $\boxed{\det(A - \lambda I) = 0}$

is called characteristics equation of A

or characteristic polynomial equation of eigenvalues of A

$k_1 = k_2 = k_3 = k$, $m_1 = m_2 = 1$

$$A = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2k - \lambda & k \\ k & -2k - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\Rightarrow (-(2k+\lambda))(-(2k+\lambda)) - k^2 = 0$$

$$\Rightarrow (2k+\lambda)^2 - k^2 = 0$$

$$\Rightarrow \lambda^2 + 4k\lambda + 4k^2 - k^2 = 0$$

$$\Rightarrow \boxed{\lambda^2 + 4k\lambda + 3k^2 = 0} \longrightarrow \text{polynomial equation of order 2.}$$

eigenvalue is a number λ that makes $A - \lambda I$ a singular matrix and therefore $(A - \lambda I)X = 0$ has non-trivial solution X.

For $n \times n$ matrix, we have $\lambda_1, \lambda_2, \dots, \lambda_n$ eigenvalues:

(i) λ's could be complex numbers

(ii) not all λ's are necessarily unique

$$I_{n\times n} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}$$

$\rightarrow$ $\lambda_1 = \lambda_2 = \dots = \lambda_n = 1$

$\rightarrow$ any possible n-dimensional vector X is eigen vector

$(I - \lambda I) X = 0$ for any x

$\downarrow$

$Ix = 1 \cdot x$

$\Rightarrow (I - 1 \cdot I) x = 0$

$$\lambda^2 + 4k\lambda + 3k^2 = 0$$

$$\Rightarrow \lambda = \frac{-4k \pm \sqrt{16k^2 - 12k^2}}{2}$$

$$= -2k \pm \tfrac{1}{2}\sqrt{4k^2}$$

$$= -2k \pm k$$

$$\boxed{\lambda = -3k, \; -k} \rightarrow \det(A - \lambda I) = 0$$

Equations for X_1, X_2

$$(A - \lambda I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = -3k$

$$\begin{bmatrix} -2k+3k & k \\ k & -2k+3k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} k & k \\ k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

two equations are same $\Rightarrow$ $k(X_1 + X_2) = 0$

$\Rightarrow$ $X_1 = -X_2$

We need additional equation :

X_1, X_2 are such that

$$\sqrt{x_1^2 + x_2^2} = 1 \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \boxed{x_1^2 + x_2^2 = 1}$$

$$\Rightarrow 2X_1^2 = 1 \Rightarrow \boxed{X_1 = \frac{1}{\sqrt{2}}}$$

$$\boxed{X_2 = -\frac{1}{\sqrt{2}}}$$

$(A - \lambda I)X = 0$ problem

$\lambda = -3k$, $X = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

If X is such that

$$AX = \lambda X \longrightarrow \boxed{\begin{array}{l}\text{eigenvector } X \\ \text{eigenvalue } \lambda\end{array}}$$

then

for any α number $Y = \alpha X$ is also eigenvector

$$AY = \lambda Y$$

$$\begin{aligned} AY = A(\alpha X) &= \alpha AX \\ &= \alpha \lambda X \\ &= \lambda(\alpha X) \\ &= \lambda Y \end{aligned}$$

$\lambda = -k$

$$\begin{bmatrix} -2k+\lambda & k \\ k & -2k+\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -k & k \\ k & -k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{array}{l} \rightarrow -x_1 + x_2 = 0 \\ \rightarrow x_1 - x_2 = 0 \end{array}$$

$0 = 0$

additional equation $\quad x_1^2 + x_2^2 = 1 \Rightarrow 2x_1^2 = 1 \Rightarrow x_1 = \frac{1}{\sqrt{2}}$

$\Rightarrow x_2 = x_1 = \frac{1}{\sqrt{2}}$

for $\lambda = -k$, $\quad X = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigenvector

so is

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $Ax = \lambda x$ problem (eigenvalue problem)

(i) $\quad \lambda = -3k, \quad x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(ii) $\quad \lambda = -k, \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -\omega^2$

$$u(t) = \sin(-\sqrt{\lambda}\, t)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \frac{d^2u}{dt^2} = Au$$

$$\left.\begin{aligned} u_1(t) &= \sin(-\sqrt{\lambda_1}\, t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ u_2(t) &= \sin(-\sqrt{\lambda_2}\, t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}\right] \rightarrow$$

Lecture 21

ODE

(1) $$\frac{d^2u}{dt^2} = Au$$

$$u(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}$$

$k_1 = k_2 = k_3 = k, \quad m_1 = m_2 = 1$

IC

$$u(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\dot{u}(0) = \frac{du}{dt}(0) = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

Find generic / incomplete solution to (1)

$$u(t) = \sin(\omega t) X, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

ω, X are some number and vector

For free

$$\boxed{\frac{d^2u}{dt^2} = -\omega^2 u}$$ for any ω, X

But we want $$\frac{d^2u}{dt^2} = Au$$

So need ω, X such that

$$\boxed{Au = -\omega^2 u}$$

$\downarrow$

If I set $\lambda = -\omega^2$

then $\boxed{Au = \lambda u}$ $\longrightarrow$ eigenvalue - eigenvector

$\downarrow$

$$A(\sin(\omega t) X) = \lambda(\sin(\omega t) X)$$

$$\Rightarrow \cancel{\sin(\omega t)}(AX) = \cancel{\sin(\omega t)}(\lambda X)$$

$\Rightarrow \boxed{Ax = \lambda X} \longrightarrow$ eigenvalue - eigenvector.

for $A = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}$

$\lambda_1 = -3k$, $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ pair 1

$\downarrow$

$-\omega_1^2 = -3k$

$\Rightarrow \omega_1 = \sqrt{3k}$

$u_1(t) = \sin(\omega_1 t) X_1$

$\Downarrow$

$\dfrac{d^2 u_1}{dt^2} = A u_1$

$\lambda_2 = -k$, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\downarrow$

$-\omega_2^2 = -k$

$\Rightarrow \omega_2 = \sqrt{k}$ pair 2

$u_2(t) = \sin(\omega_2 t) X_2$

$\Downarrow$

$\dfrac{d^2 u_2}{dt^2} = A u_2$

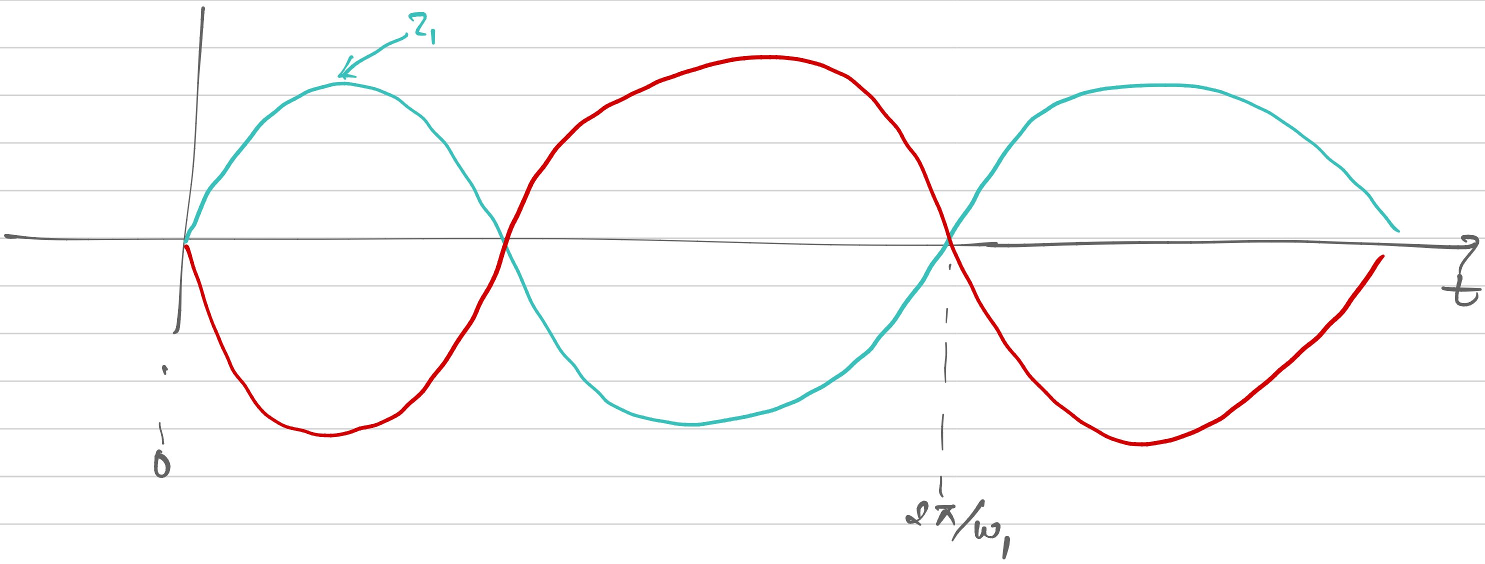
$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = u_1 = \sin(\omega_1 t)\, X_1 = \sin(\omega_1 t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$z_1(t) = \sin(\omega_1 t)$$

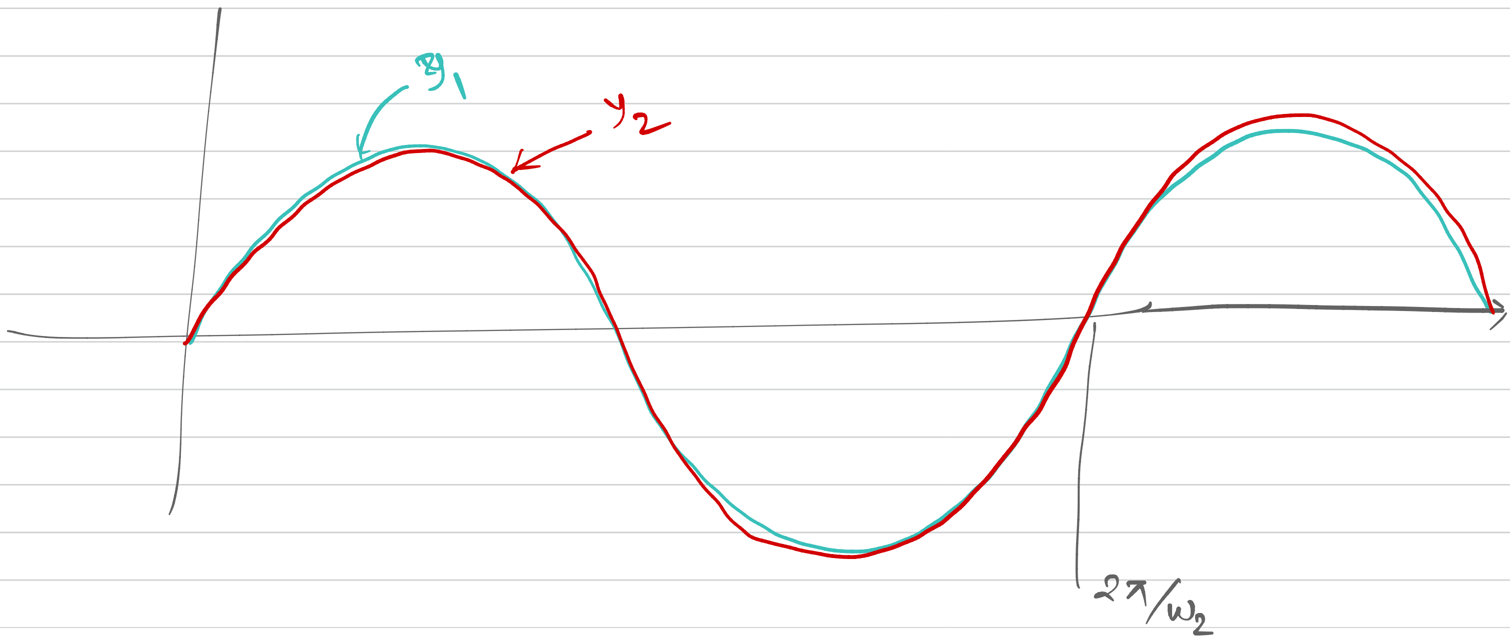
$$z_2(t) = -\sin(\omega_1 t)$$

at any

$$z_1(t) = -z_2(t)$$



$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = u_2(t) = \sin(\omega_2 t)\, X_2 = \sin(\omega_2 t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



fin α_1, α_2 numbers

$$f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \alpha_1 u_1(t) + \alpha_2 u_2(t)$$

$$\frac{d^2 f(t)}{dt^2} = \alpha_1 \frac{d^2 u_1}{dt^2} + \alpha_2 \frac{d^2 u_2}{dt^2}$$

$$= \alpha_1 A u_1 + \alpha_2 A u_2$$

$$= A(\alpha_1 u_1 + \alpha_2 u_2)$$

$$= Af$$

$\Rightarrow$ $\boxed{\frac{d^2 f}{dt^2} = Af}$ for any α_1, α_2

$f = \alpha_1 u_1 + \alpha_2 u_2$

for any α_1, α_2 numbers $f = \alpha_1 u_1 + \alpha_2 u_2$

satisfies $\frac{d^2 f}{dt^2} = Af$

find α_1 and α_2 such that

$\checkmark$ $f(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\frac{df}{dt}(0) = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$

$$f(0) = \alpha_1 u_1(0) + \alpha_2 u_2(0)$$

$$\frac{df}{dt} = \alpha_1 \frac{du_1}{dt} + \alpha_2 \frac{du_2}{dt}$$

$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= \alpha_1 \, \omega_1 \cos(\omega_1 t) \, X_1 + \alpha_2 \, \omega_2 \cos(\omega_2 t) \, X_2$$

$$\Rightarrow \frac{df}{dt}(0) = \alpha_1 \, \omega_1 \cos(0) \, X_1 + \alpha_2 \, \omega_2 \cos(0) \, X_2$$

$$= \alpha_1 \omega_1 X_1 + \alpha_2 \omega_2 X_2$$

$$\Rightarrow \frac{df}{dt}(0) = \begin{bmatrix} \alpha_1 \omega_1 + \alpha_2 \omega_2 \\ -\alpha_1 \omega_1 + \alpha_2 \omega_2 \end{bmatrix}$$

choose α_1, α_2 s.t.

$$\frac{df}{dt}(0) = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \omega_1 + \alpha_2 \omega_2 \\ -\alpha_1 \omega_1 + \alpha_2 \omega_2 \end{bmatrix}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \alpha_1 \omega_1 + \alpha_2 \omega_2 &= a_0 \\ -\alpha_1 \omega_1 + \alpha_2 \omega_2 &= b_0 \end{aligned}}$$

α_1, α_2

$$\alpha_1 = \frac{\alpha_2 \omega_2 - b_0}{\omega_1}$$

$$\alpha_2 \omega_2 - b_0 + \alpha_2 \omega_2 \omega_1 = a_0 \omega_1$$

$$\Rightarrow \alpha_2 \omega_2 (1 + \omega_1) = a_0 \omega_1 + b_0$$

$$\Rightarrow \alpha_2 = \frac{a_0 \omega_1 + b_0}{\omega_2 (1 + \omega_1)}$$

$$f(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$$

① $$\frac{d^2 f}{dt^2} = Af$$

② $$f(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \dot{f}(0) = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$