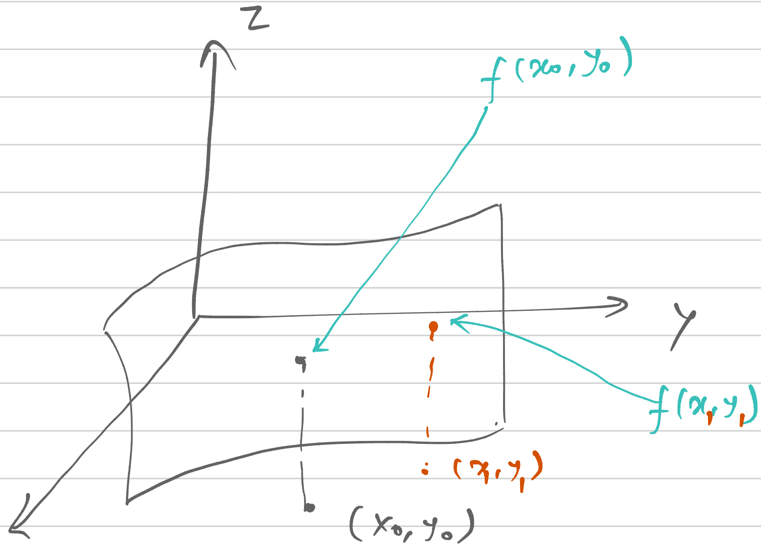
Lecture 19

Review Solving system of nonlinear equations

Taylor's series

$\hookrightarrow f = f(x, y)$

You know (x_0, y_0)



$$\boxed{f(x_1, y_1) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x_1 - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y_1 - y_0)}$$

$$= f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right] \times \underbrace{\begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}}_{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}$$

$$\frac{\partial f(x_0, y_0)}{\partial x} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$\frac{\partial f(x_0, y_0)}{\partial y} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

$$\underset{1 \times n}{a} \times \underset{n \times 1}{b} = \underset{1 \times 1}{c}$$

$$f(x_1, y_1) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right] \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \left[\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right] \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}, \quad x^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_n^1 \end{bmatrix}$$

$$f(x, y, z, \ldots, w)$$

n^{th} argument

$$f(x_1^1, x_2^1, \ldots, x_n^1) = f(x_1^0, x_2^0, \ldots, x_n^0) + \left[\frac{\partial f}{\partial x_1}(x^0), \frac{\partial f}{\partial x_2}(x^0), \ldots, \frac{\partial f}{\partial x_n}(x^0)\right] \begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_n^1 \end{bmatrix} - \left[\quad \right] \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

<u>system of nonlinear equations.</u>

$$f_1(x_1, \ldots, x_n) = 0$$

$$\vdots$$

$$f_n(x_1, \ldots, x_n) = 0$$

<u>Problem</u> $x^0 = \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix}$ ✓ , $x^1 = \begin{bmatrix} x_1^1 \\ \vdots \\ x_n^1 \end{bmatrix}$?

for any $i = 1, 2, \ldots, n$

$$0 = f_i(x^1) = f_i(x^0) + \left[\frac{\partial f_i}{\partial x_1}(x^0), \ldots, \frac{\partial f_i}{\partial x_n}(x^0)\right] \left\{ x^1 - x^0 \right\}$$

$i = 1, 2, \dots, n$

$$\underbrace{\left[\frac{\partial f_i}{\partial x_1}(x^0), \dots, \frac{\partial f_i}{\partial x_n}(x^0)\right] x'}_{} = \underbrace{-f_i(x^0) + \left[\frac{\partial f_i}{\partial x_1}(x_0), \dots, \frac{\partial f_i}{\partial x_n}(x^0)\right] x^0}_{b_i}$$

$$\boxed{a_{i_1} x_1' + a_{i_2} x_2' + \dots + a_{in} x_n' = b_i}$$

$$a_{i_1} = \frac{\partial f_i}{\partial x_1}(x^0)$$

$$a_{i_2} = \frac{\partial f_i}{\partial x_2}(x^0)$$

$$\vdots$$

$$a_{in} = \frac{\partial f_i}{\partial x_n}(x^0)$$

$$[c_1, c_2, \dots, c_n] \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = c_1 x_1' + c_2 x_2' + \dots + c_n x_n'$$

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

$$J x' = b$$

$$J(x^0) = J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^0) & \dots & \frac{\partial f_1}{\partial x_n}(x^0) \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1}(x^0) & \dots & \frac{\partial f_n}{\partial x_n}(x^0) \end{bmatrix}, \quad x' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$$

$$\frac{\partial f_2}{\partial x_2}(x^0)$$

$$b = -\begin{bmatrix} f_1(x^0) \\ \vdots \\ f_n(x^0) \end{bmatrix} + J x^0$$

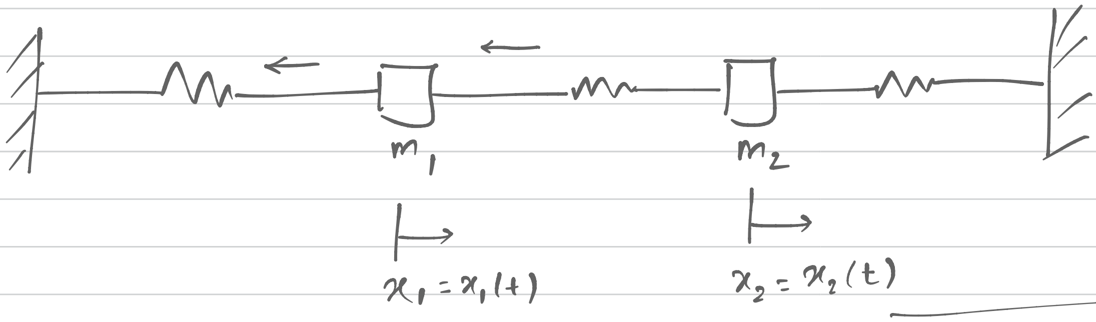
$$\Rightarrow \boxed{J x' = -\begin{bmatrix} f_1(x^0) \\ \vdots \\ f_n(x^0) \end{bmatrix} + J x^0} \Rightarrow x' \checkmark$$

x^0 ✓, $\quad x^1$ ✓, $\quad x^2$?

$$\boxed{J(x^1)\,x^2 = -\begin{bmatrix} f_1(x^1) \\ \vdots \\ f_n(x^1) \end{bmatrix} + J(x^1)\,x^1}$$

$f(x,y) \longrightarrow f(a) \qquad a = \begin{bmatrix} x \\ y \end{bmatrix}$

- Eigenvalues and eigenvectors



$$m\frac{d^2y}{dt^2} = -cy$$

$$\boxed{y = a\sin(bt)}$$

$$-ab^2\sin(bt) = \frac{d^2y}{dt^2}$$

① $\quad m_1 \dfrac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1)$

② $\quad m_2 \dfrac{d^2x_2}{dt^2} = -kx_2 - k(x_2 - x_1)$

$x_1(t) = X_1 \sin(\omega t)$ ③

$x_2(t) = X_2 \sin(\omega t)$ ④

$m_1 = 1, \; m_2 = 1$

① + ③ + ④

$$-X_1\omega^2 \sin(\omega t) = -kX_1\sin(\omega t) + k(X_2 - X_1)\sin(\omega t)$$

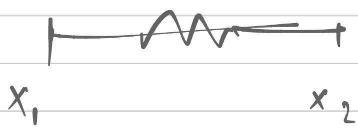
(5) $$- X_1 \omega^2 + k X_1 + k X_2 - k X_1 = 0$$

(2) + (3) + (4)

$$- X_2 \omega^2 \sin(\omega t) = - k X_2 \sin(\omega t) - k(X_2 - X_1) \sin(\omega t)$$

(6) → $$- X_2 \omega^2 + k X_2 + k X_2 - k X_1 = 0$$

$$\begin{bmatrix} -1-\omega^2 & \\ & \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$(X_{2\,new} - X_{1\,new}) - (X_2 - X_1)$$