Lecture 18

- Iterative method
- Solving nonlinear system of equations

Iterative method

(1) $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$

(2) $a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$

(3) $a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} , \qquad X^i = \begin{bmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_n^i \end{bmatrix}$$

$\Rightarrow$ $x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{bmatrix}$ is initial guess of my solution to $Ax = b$

(1) $a_{11} x_1^1 + a_{12} x_2^1 + a_{13} x_3^1 = b_1$

$$\Rightarrow \boxed{x_1^1 = \frac{b_1 - a_{12} x_2^0 - a_{13} x_3^0}{a_{11}}}$$

(2) $a_{21} x_1^1 + a_{22} x_2^1 + a_{23} x_3^1 = b_2$

x_1^0 ... x_3^0 $\longrightarrow$ Jacobi iteration

x_1^1 ... x_3^0 $\longrightarrow$ Gauss-Seidel iteration

$$\boxed{x_2^1 = \frac{b_2 - a_{21}x_1^1 - a_{23}x_3^0}{a_{22}}}$$

③ $a_{31}x_1^1 + a_{32}x_2^1 + a_{33}x_3^1 = b_3$

$$\begin{array}{cc} \downarrow & \downarrow \\ x_1^0 & x_2^0 \\ x_1^1 & x_2^1 \end{array}$$

$$\boxed{x_3^1 = \frac{b_3 - a_{31}x_1^1 - a_{32}x_2^1}{a_{33}}}$$

$$x^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \end{bmatrix}$$

$$\downarrow$$

$$x^2 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{bmatrix}$$

$$\downarrow$$

$$x^i = \begin{bmatrix} x_1^i \\ x_2^i \\ x_3^i \end{bmatrix} \longrightarrow x^{i+1} ?$$

$$x_1^{i+1} = \frac{b_1 - a_{12}x_2^i - a_{13}x_3^i}{a_{11}}$$

$$x_2^{i+1} = \frac{b_2 - a_{21}x_1^{i+1} - a_{23}x_3^i}{a_{22}}$$

$$x_3^{i+1} = \frac{b_3 - a_{31}x_1^{i+1} - a_{32}x_2^{i+1}}{a_{33}}$$

$$x_1^{i+1} = \frac{b_1 - a_{12}\, x_2^i - a_{13}\, x_3^i}{a_{11}}$$

$$x_2^{i+1} = \frac{b_2 - a_{21}\, x_1^i - a_{23}\, x_3^i}{a_{22}}$$

$$x_3^{i+1} = \frac{b_3 - a_{31}\, x_1^i - a_{32}\, x_2^i}{a_{33}}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

Given $x^i = \begin{bmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_n^i \end{bmatrix}$

$$x_1^{i+1} = \frac{b_1 - \left[a_{12}\, \overset{\overset{x_2^i}{\uparrow}}{x_2^i} + a_{13}\, \overset{\overset{x_3^i}{\uparrow}}{x_3^i} + \cdots + a_{1n}\, \overset{\overset{x_3^i}{\uparrow}}{x_n^i} \right]}{a_{11}}$$

$$\vdots$$

$$x_k^{i+1} = \frac{b_k - \left[a_{k1}\, \overset{\overset{x_1^{i+1}}{\uparrow}}{x_1^i} + a_{k2}\, \overset{\overset{x_2^{i+1}}{\uparrow}}{x_2^i} + \cdots + a_{k(k-1)}\, \overset{\overset{x_{k-1}^{i+1}}{\uparrow}}{x_{k-1}^i} + a_{k(k+1)}\, \underset{\underset{x_{k+1}^i}{\downarrow}}{x_{k+1}^i} + \cdots + a_{kn}\, \underset{\underset{x_n^i}{\downarrow}}{x_n^i} \right]}{a_{kk}}$$

$$C = \begin{bmatrix} 0 & a_{12}/a_{11} & & \cdots & a_{1n}/a_{11} \\ a_{21}/a_{22} & 0 & & \cdots & a_{2n}/a_{22} \\ a_{31}/a_{23} & a_{32}/a_{33} & 0 & \cdots & a_{3n}/a_{33} \\ \vdots & & & & \\ a_{n1}/a_{nn} & a_{n2}/a_{nn} & a_{n3}/a_{nn} & \cdots & 0 \end{bmatrix}$$

$$\bar{b} = \begin{bmatrix} b_1/a_{11} \\ b_2/a_{22} \\ \vdots \\ b_n/a_{nn} \end{bmatrix}$$

Jacobi

$$\boxed{x^{i+1} = \bar{b} - C x^i}$$

- Relaxation method

Given x^i from i^{th} iteration

$\downarrow$

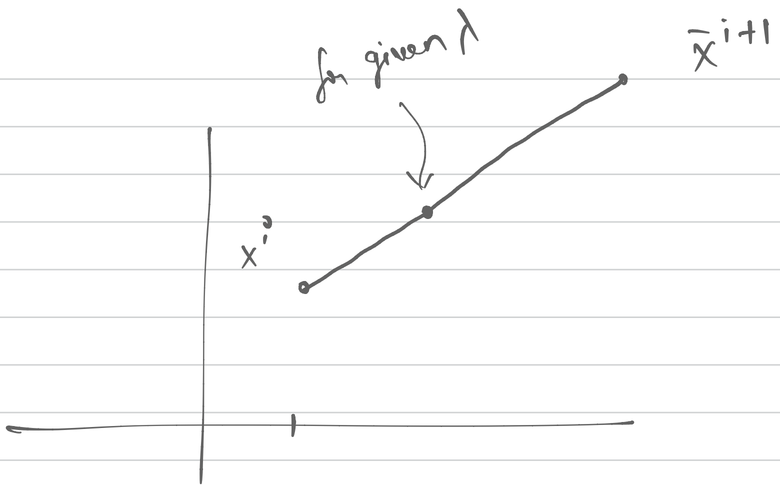
Compute $\bar{x}^{i+1}$ using Gauss-Seidel / Jacobi

$\downarrow$

$$\boxed{x^{i+1} = \lambda \bar{x}^{i+1} + (1-\lambda) x^i}$$

for pre-defined number λ, $0 < \lambda < 2$

if $\lambda \in (0,1)$ $(0 < \lambda < 1) \rightarrow$ "under relaxation"



$2 > \lambda > 1$

$\downarrow$

"Over relaxation"

- System of nonlinear equations

for 2 equation, 2 unknown
(generic form)

$$\boxed{\begin{aligned} f_1(x_1, x_2) &= 0 \\ f_2(x_1, x_2) &= 0 \end{aligned}}$$

$\longrightarrow$ find x_1, x_2 s.t this two equations are satisfied.

2 unknown, 2 equation

$f_1 = f_1(x_1, x_2) \longleftarrow \boxed{x_1 x_2^2 + x_2 - b_1 = 0}$

$f_2 = f_2(x_1, x_2) \longleftarrow \boxed{x_1^3 + x_1 x_2 - b_2 = 0}$

$\Downarrow$

Roots problem in a multivariable, multifunction setting

$$f = f(x, y) = x^2 + (\sin x)\, y^3 - 10$$

$$f(x, y) = 0$$

$g(x) = 0$, x_0, x_1 ?

$$g(x_1) = g(x_0) + (x_1 - x_0)\frac{dg}{dx}(x_0)$$

$$\Rightarrow \quad 0 = g(x_0) + (x_1 - x_0)\frac{dg}{dx}(x_0)$$

$$\Rightarrow \quad x_1 = x_0 - \frac{g(x_0)}{\frac{dg}{dx}(x_0)}$$

For our problem: $x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$, x^1 ?

$$0 = f_1(x_1^1, x_2^1) = f_1(x_1^0, x_2^0) + (x_1^1 - x_1^0)\frac{\partial f_1}{\partial x_1}(x_1^0, x_2^0) + (x_2^1 - x_2^0)\frac{\partial f}{\partial x_2}(x_1^0, x_2^0)$$

$$0 = f_2(x_1^1, x_2^1) = f_2(x_1^0, x_2^0) + (x_1^1 - x_1^0)\frac{\partial f_2}{\partial x_1}(x_1^0, x_2^0) + (x_2^1 - x_2^0)\frac{\partial f_2}{\partial x_2}(x_1^0, x_2^0)$$

$$\underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^0) & \frac{\partial f_1}{\partial x_2}(x^0) \\ \frac{\partial f_2}{\partial x_1}(x^0) & \frac{\partial f_2}{\partial x_2}(x^0) \end{bmatrix}}_{J(x^0)} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = -\begin{bmatrix} f_1(x^0) \\ f_2(x^0) \end{bmatrix} + J(x^0)\begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$$

$$\Rightarrow \boxed{Ax = b}$$

$$x = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}, \quad A = J(x^0)$$

$$b = -\begin{bmatrix} f_1(x^0) \\ f_2(x^0) \end{bmatrix} + J(x^0)\begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$$

2^{nd} step

$Ax = b$

$$x = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}, \quad A = J(x^1), \quad b = -\begin{bmatrix} f_1(x^1) \\ f_2(x^1) \end{bmatrix} + J(x^1)\, x^1$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boxed{\begin{aligned} f_1(x) &= 0 \\ f_2(x) &= 0 \\ &\vdots \\ f_n(x) &= 0 \end{aligned}}$$

$$\Downarrow$$

$$\boxed{J(x^0)\, x^1 = -\begin{bmatrix} f_1(x^0) \\ \vdots \\ f_n(x^0) \end{bmatrix} + J(x^0)\, x^0}$$