# Lecture 17

Topic 4
- Matrix notations
- Methods to solve $Ax = b$
- LU factorization of a matrix A
- Iterative (numerical) methods for $Ax = b$
- Inverse $A^{-1}$, condition number of matrix

Topic 5
- Eigenvalues and eigenvectors

## Types of matrices

- Square matrix  # rows = # columns
- Symmetric matrix  $A^T = A$  (square matrix)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\Downarrow$$

$$a_{ji} = a_{ij}$$

- Skew-symmetric matrix  $A^T = -A$

$$\Downarrow$$

$$a_{ji} = -a_{ij}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$a_{21} = -a_{12}$

$a_{31} = -a_{13}$, $a_{32} = -a_{23}$

Any square matrix A has following additive unique decomposition

$$\boxed{A = A_{sym} + A_{skew}}$$

$A_{sym} = \frac{1}{2}(A + A^T) \longrightarrow$

$A_{skew} = \frac{1}{2}(A - A^T) \longrightarrow$

- Triangular matrix

  - upper triangular matrix (square)

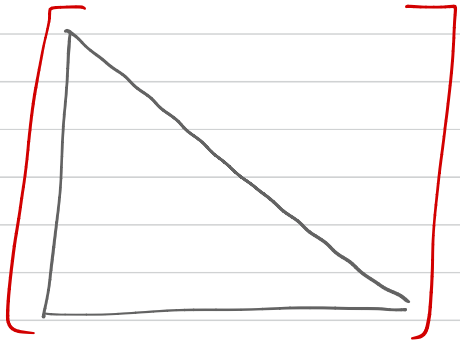
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ 0 & 0 & 0 & \ddots & a_{(n-1)n} \\ 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

$A_{n \times n}$ then A is upper triangle if, for any $j = 1, 2, \ldots, n$,

$$a_{ij} = 0 \quad \text{for all } i > j$$

- lower triangular matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & 0 \\ a_{n1} & a_{n2} & & & a_{nn} \end{bmatrix}$$



$A_{n\times n}$ is lower triangle matrix if for any $j = 1, 2, \ldots, n$

$$a_{ij} = 0 \quad \text{for all} \quad i < j$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \underbrace{A_{upper} + A_{lower} + A_{diagonal}}_{\text{unique decomposition}}$$

$$= \underbrace{A_{upper} + A_{lower}}_{\text{not unique}}$$

- Diagonal matrix

$A_{n\times n}$ is called diagonal matrix

if for any $i, j$, such that $i \neq j$,

$$a_{ij} = 0$$

# LU factorization of a matrix

Given a square matrix $A_{n\times n}$, and upper triangle matrix $U_{n\times n}$, lower triangle matrix $L_{n\times n}$.

We call $U$ and $L$, LU factorization of matrix $A$ if

$$A = LU$$

- LU decomposition is possible for invertible square matrix.

## Gauss-elimination method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\Downarrow$ 1st step of forward elimination

$$f_{21} = \frac{a_{21}}{a_{11}} \qquad f_{31} = \frac{a_{31}}{a_{11}}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_2^{(2)} \end{bmatrix}$$

$\downarrow$ 2nd step

$$f_{32} = \frac{a_{32}^{(1)}}{a_{22}^{(1)}}$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{23}^{(2)} \end{bmatrix}}_{U}, \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \end{bmatrix}$$

<u>define L as follows</u>

$$L = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{23}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \cancel{f_{21}\, a_{11}}\ a_{21} & \cancel{f_{21}\, a_{12} + a_{22}^{(1)}}\ a_{22} & a_{22} \\ a_{31} & a_{32} & f_{31}\, a_{13} + f_{32}\, a_{23}^{(1)} + \phi\, a_{33}^{(2)} \quad a_{33} \end{bmatrix}$$

- $f_{21} \, a_{11} = \frac{a_{21}}{a_{11}} a_{11} = a_{21}$

- $f_{21} \, a_{12} + a_{22}^{(1)} = \cancel{\frac{a_{21}}{a_{11}} a_{12}} + a_{22} - \cancel{\frac{a_{21}}{a_{11}} \times a_{12}}$

$$= a_{22}$$

- $f_{31} \, a_{13} + f_{32} \, a_{23}^{(1)} + a_{33}^{(2)}$

$$= \cancel{\underbrace{\frac{a_{31}}{a_{11}}}_{f_{31}} a_{13}} + \cancel{\underbrace{\frac{a_{32}^{(1)}}{a_{22}^{(1)}}}_{f_{32}} a_{23}^{(1)}} + \underbrace{a_{33}^{(1)}}_{a_{33} - \cancel{f_{31} \, a_{13}}} - \cancel{f_{32} \, a_{23}^{(1)}}$$

$$= a_{33}$$

## Solve $Ax = b$ using LU factorization

$$Ax = b$$

$$\Rightarrow LUx = b \implies \boxed{d = Ux \, , \quad Ld = b}$$

Let me define $d$ as follows

$$Ld = b$$

$$\boxed{\begin{aligned} Ld &= b \\ Ux &= d \end{aligned}}$$

$$Ld = b$$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{aligned} & l_{11} d_1 = b_1 \\ & l_{21} d_1 + l_{22} d_2 = b_2 \\ & l_{31} d_1 + l_{32} d_2 + l_{33} d_3 = b_3 \end{aligned}$$

$$\Rightarrow \quad d_1 = \frac{b_1}{l_{11}}$$

$$d_2 = \frac{b_2 - d_1 l_{21}}{l_{22}}$$

$$d_3 = \frac{b_3 - d_1 l_{31} - d_2 l_{32}}{l_{33}}$$

$$Ux = d$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \rightarrow \begin{aligned} & u_{11} x_1 + u_{12} x_2 + u_{13} x_3 = d_1 \\ & u_{22} x_2 + u_{23} x_3 = d_2 \\ & u_{33} x_3 = d_3 \end{aligned}$$

$$\Rightarrow \quad x_3 = \frac{d_3}{u_{33}}$$

$$\Rightarrow \quad x_2 = \frac{d_2 - x_3 u_{23}}{u_{22}}$$

$$\Rightarrow \quad x_1 = \frac{d_1 - x_2 u_{12} - x_3 u_{13}}{u_{11}}$$

## Cholesky factorization   Symmetric matrix A

$$A = U^T U$$

U is a upper triangle matrix.