Lecture 10

- Recap "Open methods" for roots problem

  - fixed-point method $f(x)=0 \Rightarrow \boxed{x=g(x)}$

    $$x^i = g(x^{i-1})$$

    where $g(x) = x - f(x)$

  - Newton-Raphson method

    $$x^i = x^{i-1} - \frac{f(x^{i-1})}{f'(x^{i-1})}$$

  - Extension of Newton-Raphso method called Secant's method

Optimization problem (Minimization)

Given $J : X \to (-\infty, \infty)$,

find $x \in X$ such that

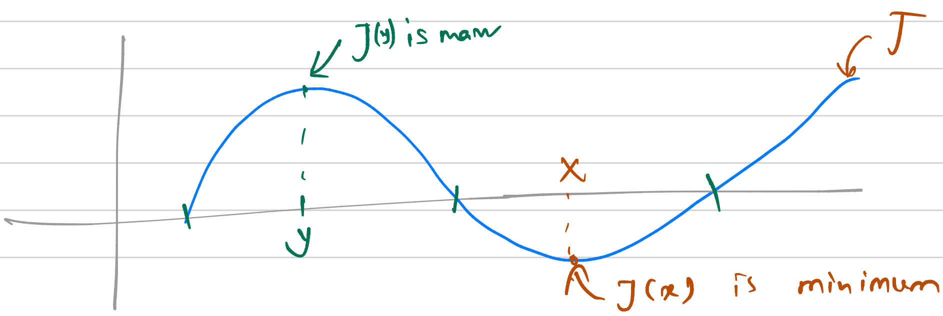
$$J(x) \leq J(y) \quad \text{for all } y \in X$$

$$\Downarrow$$

$$\min_{x \in X} J(x)$$

$$\Downarrow$$

find $x \in X$ such that $J(x)$ is minimum.



$$\min_{x \in X} J(x) \iff \max_{x \in X} -J(x)$$

$$\Updownarrow$$

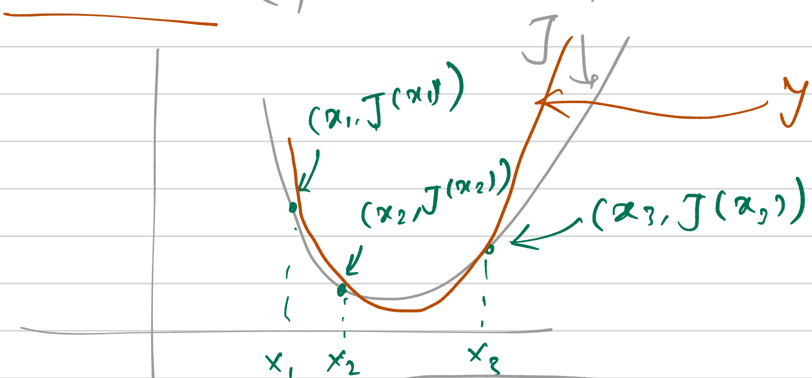
find $x \in X$ such that

$$J(x) \geq J(y) \quad \text{for all } y \in X$$

$$f(x, y) = \sin(x)^2 + \cos(x)\sin(y)$$

Numerical methods for optimization problem of single variable functions

- Golden-Section method
- Quadratic (parabola) interpolation method



$$y(x) = ax^2 + bx + c$$

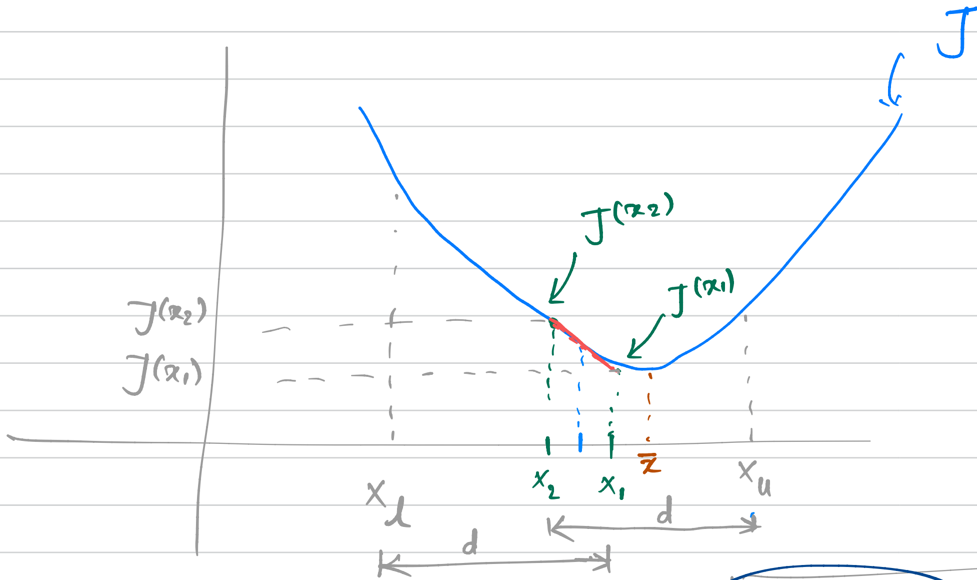
find a, b, c

$$\frac{dy}{dx} = 0 \Longleftarrow$$

$$\Rightarrow 2ax + b = 0$$

$$\Rightarrow \boxed{x = -b/2a}$$

Golden-Section method



Problem

find $\bar{x} \in [x_l, x_u]$ such that

$J(\bar{x}) \leq J(x)$

for all $x \in [x_l, x_u]$

$x_1 = x_l + d$

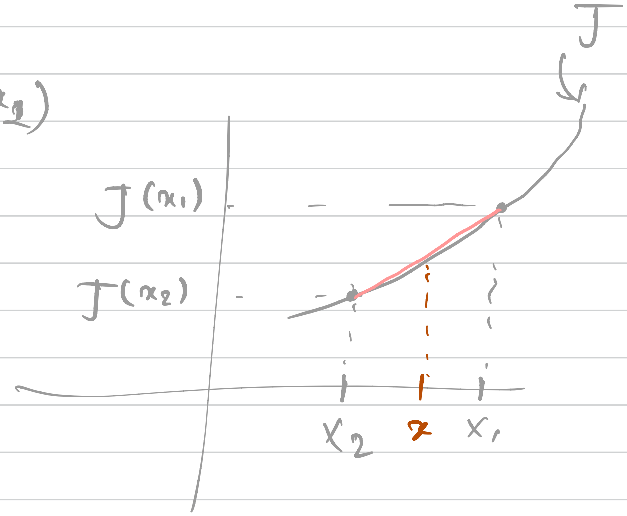
$x_2 = x_u - d$

$x_1 > x_2$ d such that

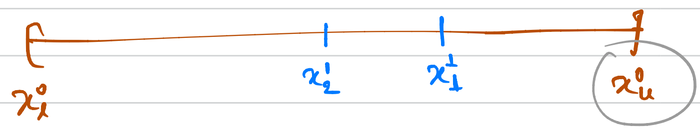
Case 1: If $J(x_1) < J(x_2)$

Take $[x_2, x_u]$ interval to search for $\bar{x}$

Case 2: If $J(x_1) > J(x_2)$



first iteration



$$x_1^1 = x_l^0 + d \qquad \text{let's say} \quad d = \alpha\,\overbrace{(x_u^0 - x_l^0)}^{\delta}$$

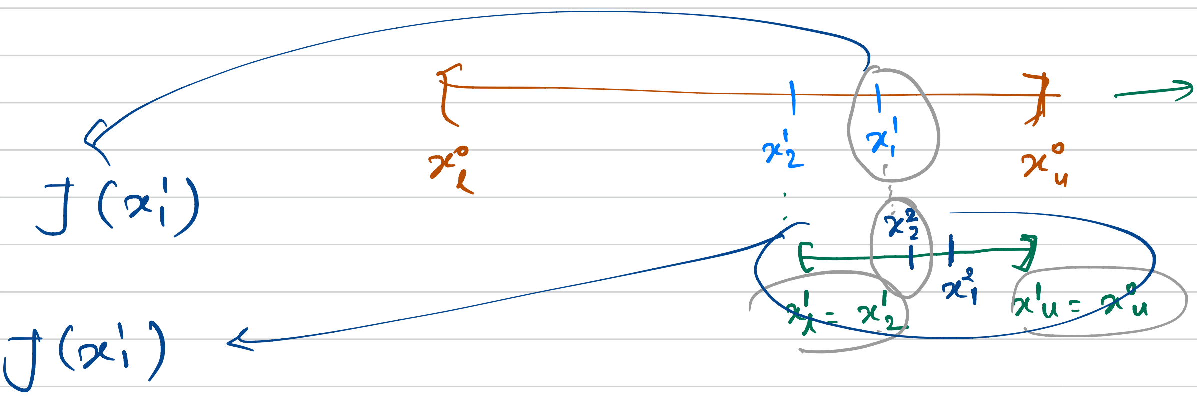
$$x_2^1 = x_u^0 - d \qquad \boxed{d = \alpha\,\delta}$$

$$\delta = x_u^0 - x_l^0$$

Suppose for second iteration, I choose

$[x_2^1, x_u^0]$

as the new interval



x_2^1, x_1^1 ✓

x_l^1, x_u^1 ✓

$$x_1^2 = x_l^1 + d = x_l^1 + \alpha(x_u^1 - x_l^1)$$

$$x_2^2 = x_u^1 - d = x_u^1 - \alpha(x_u^1 - x_l^1)$$

To find α such that

$$x_1^1 = x_2^2$$

$$\Rightarrow \quad x_l^0 + \alpha\,\delta = x_u^1 - \alpha(x_u^1 - x_l^1)$$

$$= x_u^0 - \alpha(\cancel{x_u^0} - \cancel{x_u^0} + \alpha\,\delta)$$

$$\Rightarrow \quad x_l^0 + \alpha\cdot\delta = x_u^0 - \alpha^2 \delta$$

$$\Rightarrow \quad \alpha^2 \delta + \alpha \delta + \underbrace{x_l^0 - x_u^0}_{-\delta} = 0$$

$$\Rightarrow \quad \boxed{\alpha^2 \delta + \alpha \delta - \delta = 0}$$

$$\alpha = \phi - 1$$

golden ratio

$$\boxed{\phi^2 - \phi - 1 = 0}$$