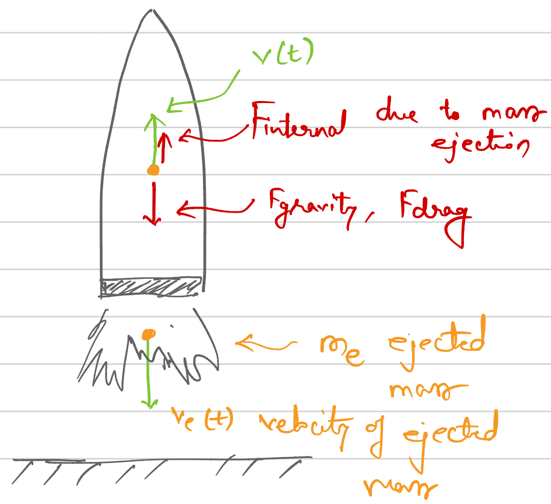
Rocket Equation

- Consider a rocket with initial mass $m(0)$ (kg)



- Rocket is ejecting mass m_e at a prescribed rate $\dot{m}_e$ (kg/s) at a relative velocity (relative to the rocket) V_e (m/s)

- Let $v = v(t)$, $m = m(t)$ are velocity (upward) (m/s) and mass of rocket (kg) at time $t=0$.

- Assume
    - $\dot{m}_e = \dot{m}_e(t)$ is known $\longrightarrow$ Notatation $\dot{f}$ means $\frac{df}{dt}$
    - $v(0) = 0$ is given
    - $m(0) = m_0$ is known
    - $g = 9.81$ (m/s²) gravity acceleration
    - C_d drag coefficient (m/kg) is also known
    - $V_e \leq 0$ relative velocity (m/s) of mass ejecting (negative as relative velocity is downward)

- Objective

Find velocity $v = v(t)$ of rocket at any time t, $0 \leq t \leq t_F$

Method

1. Conservation of mass as rocket is changing its mass

2. Conservation of linear momentum as velocity of rocket (and momentum) is changing in time due to mass ejection

(A.) Conservation of mass

Rate of change of mass of rocket = − Rate of mass out of rocket + Rate of mass into rocket

$$\Rightarrow \quad \frac{dm}{dt} = \underbrace{-\frac{dm_e}{dt}}_{\text{negative as mass goes out of rocket}} + 0$$

↑ In our problem, we are not adding any mass to rocket

Using notation $\dot{f} = \frac{df}{dt}$, we have

(1) ——— $\boxed{\dot{m}(t) = -\dot{m}_e(t)}$

(B.) Conservation of linear momentum

Rate of change of linear momentum = Internal force due to mass ejection + External force

Internal force due to mass ejection: mass ejecting applies force on rocket, $F_{internal}$

External force: gravity, $F_{gravity}$; drag, F_{drag}

- Linear momentum $p = p(t) = m(t)\, v(t)$

$$\Rightarrow \quad \frac{dp}{dt} = \frac{d}{dt}(mv) = v\frac{dm}{dt} + m\frac{dv}{dt}$$

product rule

- $F_{gravity} = -mg$ (Our convention is upward +ve, downward −ve)

- $F_{drag} = -C_d\, |v|\, v$ (drag always opposite to velocity)

- $F_{internal} = -\dot{m}_e v_e$

Note: v_e is negative as it is downward relative to rocket

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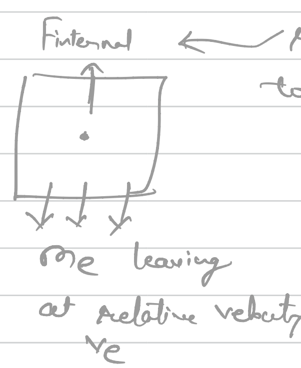
$F_{internal} \geq 0$ for $v_e \leq 0$



Thus from conservation of linear momentum law

$$m\frac{dv}{dt} + v\frac{dm}{dt} = -\dot{m}_e v_e - mg - C_d\, |v|\, v$$

$\because \quad \frac{dm}{dt} = -\dot{m}_e$ (from conservation of mass)

we have

(2) — $$\boxed{m\frac{dv}{dt} = \dot{m}_e(v - v_e) - mg - C_d|v|v}$$

OR if at time t, $m(t) > 0$, then

(3) — $$\boxed{\frac{dv}{dt} = \frac{\dot{m}_e}{m}(v - v_e) - g - \frac{C_d}{m}|v|v}$$

Let m_0 (kg), v_0 (m/s), v_e (m/s), g (m/s²), C_d (kg/m) and $\dot{m}_e(t)$ function is given.

Then we can compute $m(t)$, $v(t)$ at time t, $0 < t \leq t_F$, using

1. $m(0) = m_0$, $v(0) = v_0$

2. $\dot{m}(t) = -\dot{m}_e(t)$

3. $\frac{dv}{dt} = \frac{\dot{m}_e}{m}(v - v_e) - g - \frac{C_d}{m}|v|v$