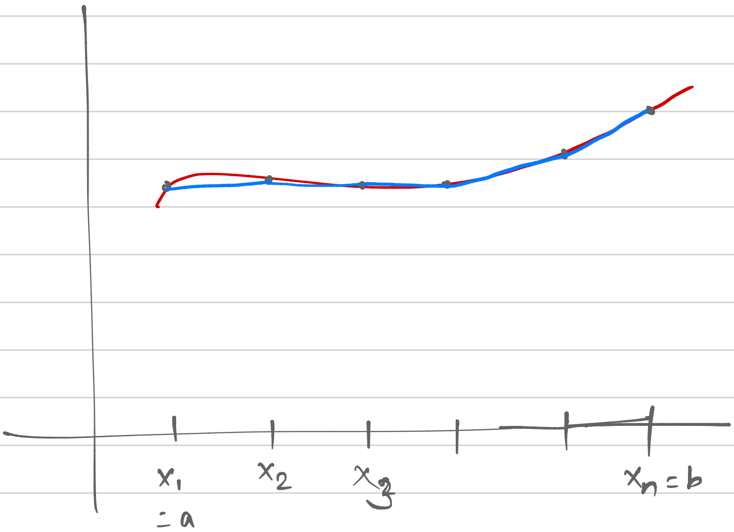
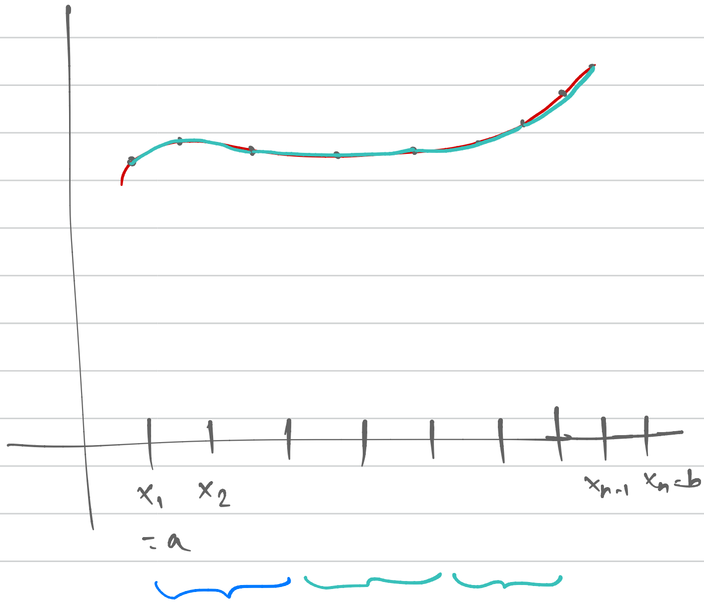
Lecture 30

- piecewise interpolation



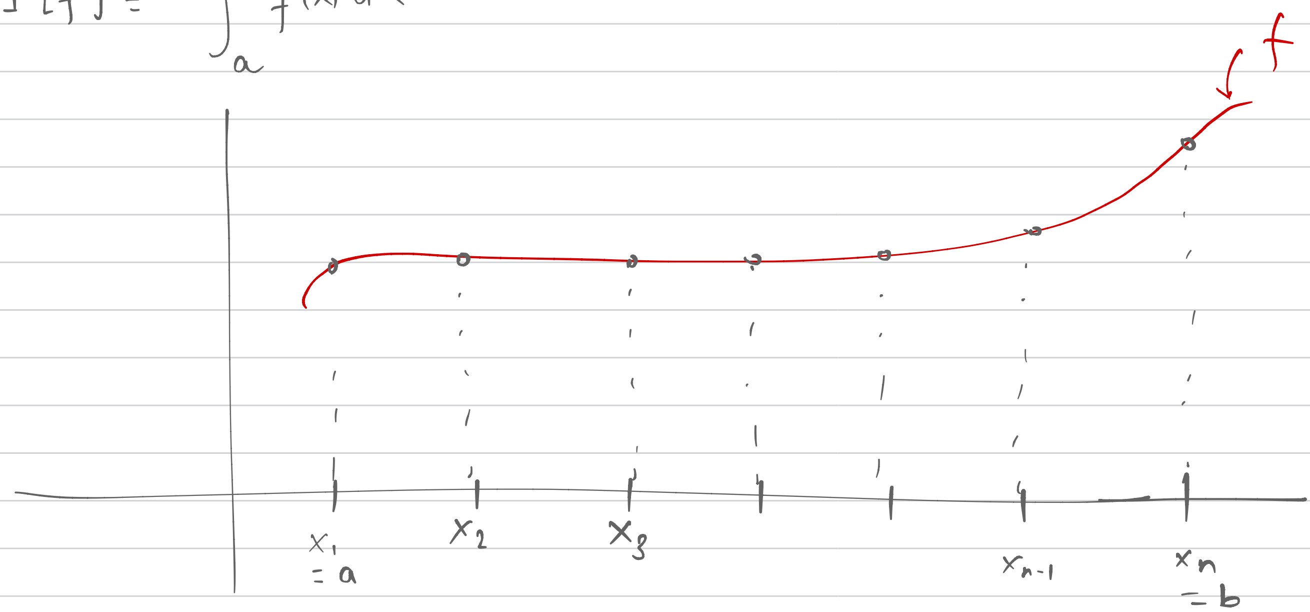


- Numerical integration

$$f : [a, b] \longrightarrow (-\infty, \infty)$$

$$(x_1, y_1 = f(x_1)), \ (x_2, y_2 = f(x_2)), \ (x_3, y_3 = f(x_3)), \dots, (x_n, y_n = f(x_n))$$

$$I[f] = \int_a^b f(x)\, dx$$

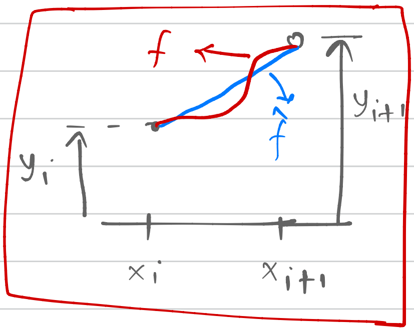


Linear interpolation

$$I[f] = \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

(x_i, y_i), (x_{i+1}, y_{i+1}) → fit a line $\hat{f}$

$\hat{f}$ will approximate f in $[x_i, x_{i+1}]$



$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \int_{x_i}^{x_{i+1}} \hat{f}(x)\,dx$$

$$= \int_{x_i}^{x_{i+1}} \left[y_i \left(\frac{x - x_{i+1}}{x_i - x_{i+1}} \right) + y_{i+1} \left(\frac{x - x_i}{x_{i+1} - x_i} \right) \right] dx$$

$$= \frac{(x_{i+1} - x_i)}{2} \left[y_i + y_{i+1} \right]$$

$$I[f] \approx \left(\frac{x_2 - x_1}{2} \right) [y_1 + y_2] + \left(\frac{x_3 - x_2}{2} \right) (y_2 + y_3) + \cdots + \frac{(x_n - x_{n-1})}{2} (y_{n-1} + y_n)$$

General Trapezoidal rule

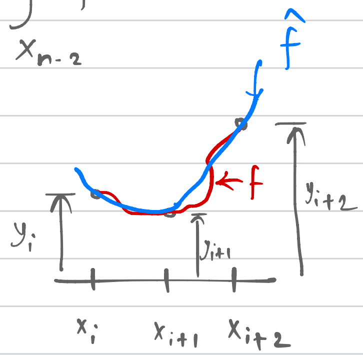
If we further assume, $x_{i+1} - x_i = h$ $\forall\ i = 1, 2, \ldots, n-1$

$$I[f] \approx \frac{h}{2} \left[y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n \right]$$

Quadratic interpolation

$$I[f] = \int_{x_1}^{x_3} f(x)\,dx + \int_{x_3}^{x_5} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}) → fit a quadratic curve $\hat{f}$



$\hat{f}$ will approximate f in $[x_i, x_{i+2}]$

$$\int_{x_i}^{x_{i+2}} f(x)\,dx \approx \int_{x_i}^{x_{i+2}} \hat{f}(x)\,dx$$

$$= \int_{x_i}^{x_{i+2}} \left[y_i \frac{(x - x_{i+1})(x - x_{i+2})}{(x_i - x_{i+1})(x_i - x_{i+2})} + \cdots \right] dx$$

assume that

$$x_{i+1} = \frac{x_i + x_{i+2}}{2}$$

$$h = \underbrace{x_{i+1} - x_i}_{} = \underbrace{x_{i+2} - x_{i+1}}_{h} = \frac{(x_{i+2} - x_i)}{2}$$

$$\boxed{\int_{x_i}^{x_{i+2}} \hat{f}\,dx = \frac{h}{3} \left[y_i + 4y_{i+1} + y_{i+2} \right]}$$

Simpson's $\frac{1}{3}$rd rule

$$= \frac{(x_{i+2} - x_i)}{6} \left[y_i + 4y_{i+1} + y_{i+2} \right]$$

if $x_2 = \frac{x_1 + x_3}{2}$, $x_4 = \frac{x_3 + x_5}{2}$, $\dots$, $x_{n-1} = \frac{x_{n-2} + x_n}{2}$

$$I[f] \approx \frac{(x_3 - x_1)}{6}\left[y_1 + 4y_2 + y_3\right] + \frac{(x_5 - x_3)}{2}\left[y_3 + 4y_4 + y_5\right]$$

$$+ \dots + \frac{(x_n - x_{n-2})}{2}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

add to that

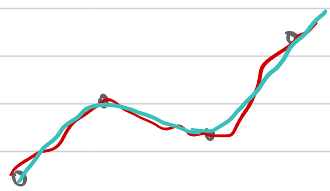
$$x_2 - x_1 = x_3 - x_2 = \dots = x_n - x_{n-1} = h$$

$$I[f] = \frac{h}{6}\left[y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \dots + 4y_{n-1} + y_n\right]$$

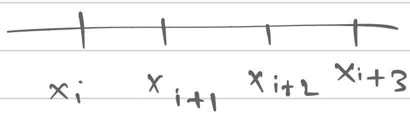
- Cubic interpolation to get approximation

$$I[f] = \int_{x_1}^{x_4} f(x)\,dx + \int_{x_4}^{x_7} f(x)\,dx + \dots + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

<u>pick</u> $[x_i, x_{i+3}]$



(x_i, y_i)
(x_{i+1}, y_{i+1})
(x_{i+2}, y_{i+2})
(x_{i+3}, y_{i+3})
$\rightarrow$ fit a cubic function $\hat{f}$



assume that $x_{i+1} - x_i = x_{i+2} - x_{i+1} = x_{i+3} - x_{i+2} = h$

$$\int_{x_i}^{x_{i+3}} f(x)\,dx \approx \int_{x_i}^{x_{i+3}} \hat{f}(x)\,dx = \frac{3h}{8}\left[y_i + 3y_{i+1} + 3y_{i+2} + y_{i+3}\right]$$

Simpson's $\frac{3}{8}$th rule

$$I[f] \approx \frac{3h}{8}\left[y_1 + 3y_2 + 3y_3 + 2y_4 + 3y_5 + 3y_6 + 2y_7 + \cdots + 3y_{n-1} + y_n \right]$$

with $x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1} = h$.

- Error in approximation of integral using piecewise polynomial

- Linear interpolation

$$I[f] = \int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_1 + 2y_2 + \cdots + y_n \right] = \hat{I}[f]$$

$$E_t = I[f] - \hat{I}[f]$$

$$\boxed{E_t = -\frac{(b-a)^3}{12n^2}\bar{f}^{(2)}}$$

$$f^{(k)} = \frac{d^k f}{dx^k}$$

$$\bar{f}^{(k)} = \frac{1}{n}\sum_{i=1}^{n-1} f^{(k)}(\eta_i)$$

where η_i is some point in $[x_i, x_{i+1}]$

- $f = \text{const}$
- $f = \text{linear}$

$\Big\}\ E_t = 0$

- $f = \text{quadratic}$ $\quad |E_t| = C\frac{1}{n^2}$

Some M s.t. $|\bar{f}^{(k)}| \le M$

behavior of errors

$n \to \infty$, $E_t \to 0$

at a quadratic speed $\frac{1}{n^2} \to 0$

it is higher compared to $\frac{1}{n} \to 0$

- Cubic interpolation (Simpson's $\frac{3}{8}$ rule)

$$E_t = -\frac{C\,(b-a)^5}{n^4}\,\bar{f}^{(4)}$$

- $f = \text{const}$, linear, quadratic, cubic, $E_t = 0$

- $n \longrightarrow \infty$, $E_t \to 0$ at a speed $\frac{1}{n^4}$