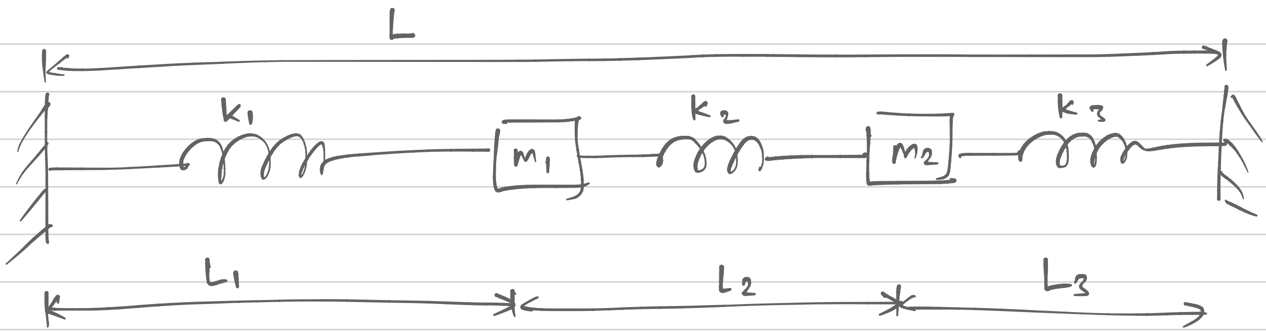
Lecture 20



at time t

Initial condition

$x_1(0) = L_1$

$x_2(0) = L_1 + L_2$

position

velocity

$\dot{x}_1(0) = v_0$

$\dot{x}_2(0) = w_0$

x_1 and x_2 satisfy

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_2)$$

$$m_2 \frac{d^2 x_2}{dt^2} = k_3(L_1 + L_2 - x_2) - k_2(x_2 - x_1 - L_2)$$

$x_2 - x_1 - L_2$
$= y_2 + (\cancel{L_1} + \cancel{L_2})$
$- (y_1 + \cancel{L_1})$
$- \cancel{L_2}$
$= y_2 - y_1$

Change of variable

$y_1 = x_1 - L_1$

$y_2 = x_2 - (L_1 + L_2)$

$$\frac{dy_1}{dt} = \frac{dx_1}{dt}, \quad \frac{d^2 y_1}{dt^2} = \frac{d^2 x_1}{dt^2}$$

$$\frac{dy_2}{dt} = \frac{dx_2}{dt}, \quad \frac{d^2 y_2}{dt^2} = \frac{d^2 x_1}{dt^2}$$

Initial condition

$y_1(0) = 0, \quad y_2(0) = 0$

$\dot{y}_1(0) = v_0, \qquad \dot{y}_2(0) = w_0$

ODEs will transform to

$$m_1 \frac{d^2 y_1}{dt^2} = -k_1 y_1 + k_2 (y_2 - y_1)$$

$$m_2 \frac{d^2 y_2}{dt^2} = -k_3 y_2 - k_2 (y_2 - y_1)$$

$$\Rightarrow \begin{cases} \dfrac{d^2 y_1}{dt^2} = -\dfrac{k_1}{m_1} y_1 + \dfrac{k_2}{m_1}(y_2 - y_1) \\ \dfrac{d^2 y_2}{dt^2} = -\dfrac{k_3}{m_2} y_2 - \dfrac{k_2}{m_2}(y_2 - y_1) \end{cases}$$

Vector - matrix notation

$$u = u(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{k_1}{m_1} - \frac{k_2}{m_1} & k_2/m_1 \\ \frac{k_2}{m_2} & -\frac{k_3}{m_2} - \frac{k_2}{m_2} \end{bmatrix}$$

$$\dot{u} = \frac{du}{dt} = \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix}, \quad \frac{d^2 u}{dt^2} = \begin{bmatrix} \frac{d^2 y_1}{dt^2} \\ \frac{d^2 y_2}{dt^2} \end{bmatrix} = \ddot{u}$$

$$\boxed{\frac{d^2 u}{dt^2} = A u}$$

$$u(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \dot{u}(0) = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$$

$u = u(t)$, $\boxed{\dfrac{d^2u}{dt^2} = -a u}$, $u(0) = 0$, $\dot{u}(0) = u_0$

$\downarrow$

$u(t) = \alpha \sin(\beta t)$

$u(0) = 0$ ✓

$\dot{u}(0) = \alpha\sqrt{a}\cos(0) = \alpha\sqrt{a}$

$\alpha\sqrt{a} = u_0$

$\alpha = \dfrac{u_0}{\sqrt{a}}$

$$\frac{du}{dt} = \alpha \beta \cos(\beta t)$$

$$\boxed{\frac{d^2u}{dt^2} = -\beta^2 \alpha \sin(\beta t) = -\beta^2 u}$$

So as long as $\beta = \sqrt{a}$

and $u(t) = \alpha \sin(\sqrt{a}\, t)$

$\downarrow$

$$\boxed{\frac{d^2u}{dt^2} = -a u}$$

we have

$$\frac{d^2u}{dt^2} = A u$$

Define $u(t) = \sin(\omega t) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ here ω, X_1, X_2 are numbers (constant)

we want $u(t)$ to be such that

$$\frac{d^2u}{dt^2} = A u$$

$$\frac{du}{dt} = \omega \cos(\omega t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \quad \frac{d^2u}{dt^2} = -\omega^2 \sin(\omega t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= -\omega^2 u$$

$$\frac{d^2u}{dt^2} = \underbrace{-\omega^2 u \quad = \quad A u}_{\text{want}}$$

$$\Downarrow$$

$$\Rightarrow \underbrace{\begin{bmatrix} -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{(k_2+k_3)}{m_2} \end{bmatrix}}_{A} \left(\sin(\omega t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

$$= (-\omega^2) \sin(\omega t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \cancel{\sin(\omega t)} \; A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cancel{\sin(\omega t)} \left(-\omega^2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

$$\Rightarrow A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\omega^2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

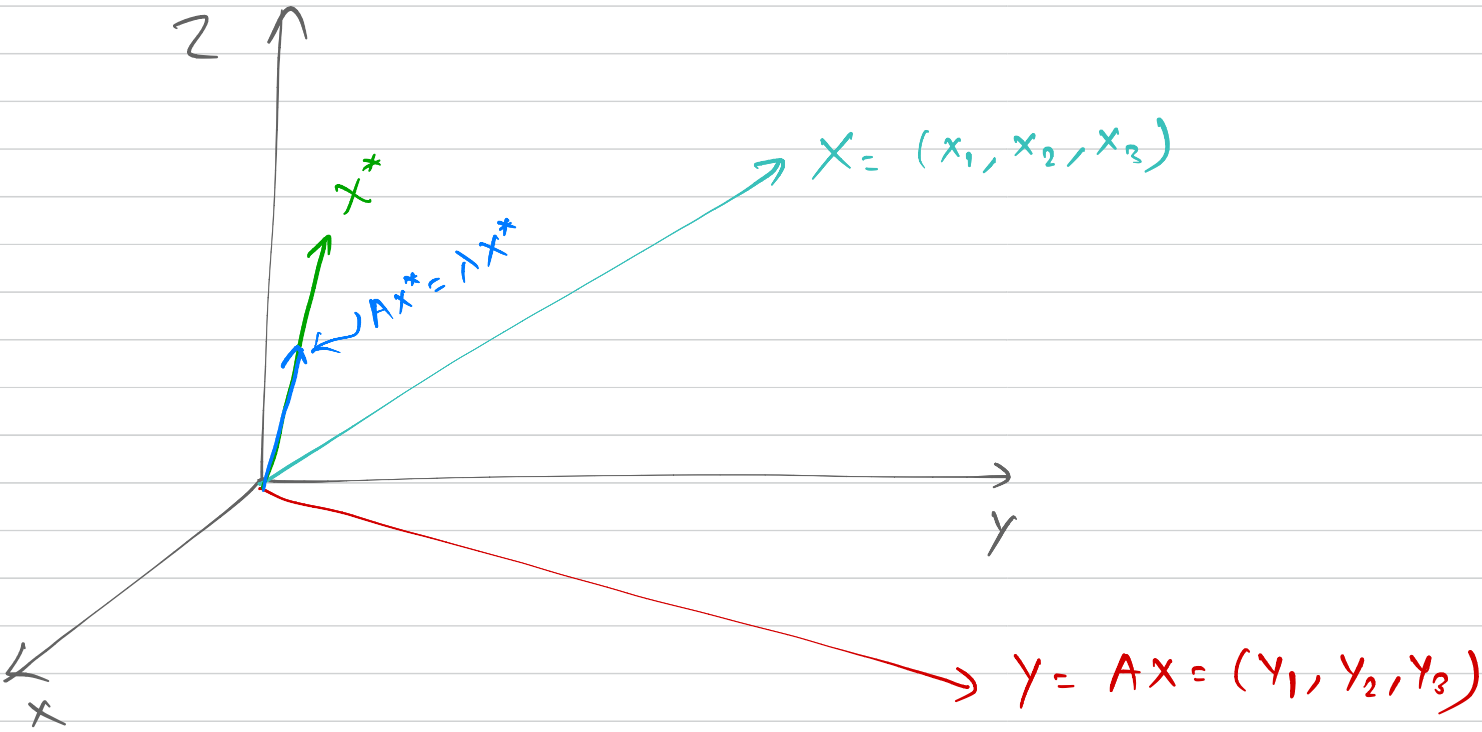
$$A(\alpha x) = \alpha A x$$

<u>Introduce</u> $\lambda = -\omega^2$

$$\Rightarrow \boxed{AX = \lambda X,} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

eigenvalue - eigenvector problem

and (λ, X) that satisfy eigenvalue problem are called eigenvalue and eigenvector, respectively.



In eigenvalue problem we are asking for λ, X s.t

$$Y = AX = \lambda X$$

(λ, x) are specials pairs and not any number and any vector will satisfy $Ax = \lambda X$

How do we find λ and X s.t $AX = \lambda X$

(1) $AX = \lambda X \Rightarrow$

$$a_{11} x_1 + a_{12} x_2 = \lambda x_1$$
$$a_{21} x_1 + a_{22} x_2 = \lambda x_2$$

(2) new observation

$$AX = \lambda X \Rightarrow (A - \lambda I) X = 0$$

$(\lambda I X = \lambda X$

$$\Downarrow$$

$$B = A - \lambda I$$

$$BX = 0$$

(i) $BX = 0$ is satisfied if $X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(ii) we are not interested in trivial solution of $BX = 0$

(iii) for any matrix B, if there is X such that

$BX = 0$ and X is not a zero vector

then

B is singular or in other words

$$\det(B) = 0$$

(iv) find λ such that $B = A - \lambda I$ is singular

$$\Downarrow$$

$$\det(B) = 0$$

$$\Rightarrow \boxed{\det(A - \lambda I) = 0}$$

$$A - \lambda I = \begin{bmatrix} -\frac{(k_1 + k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{(k_2 + k_3)}{m_2} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$k_1 = k_2 = k_3 = k$, $\quad m_1 = m_2 = 1$

$$\Rightarrow \quad A - \lambda I = \begin{bmatrix} -2k-\lambda & k \\ k & -2k-\lambda \end{bmatrix}$$

find λ s.t.

$$\det(A-\lambda I) = 0$$

$$\Rightarrow \quad (2k+\lambda)^2 - k^2 = 0$$

$$\Rightarrow \quad \boxed{\lambda^2 + 4k\lambda + 3k^2 = 0}$$

Ansn $\det(A-\lambda I) = 0 \longrightarrow$ have $\lambda_1, \lambda_2, \ldots, \lambda_n$

$\downarrow$

Characteristic equation for eigenvalues

Characteristic polynomials

$\downarrow$

Order of polynomial = size of matrix A

= # rows = # columns

(i) not all eigenvalues need to be real numbers

(ii) not all eigenvalues need to be different

$$\lambda = \frac{-4k \pm \sqrt{16k^2 - 12k^2}}{2}$$

$$= -2k \pm \frac{1}{2}\sqrt{4k^2} = -2k \pm k$$

$$\Rightarrow \quad \boxed{\lambda = -3k, \; -k}$$

Next, solve for X

$$AX = \lambda X$$

(i) $\lambda = -k$

$$AX = -kX \quad \Rightarrow \quad (A + kI)X = 0$$

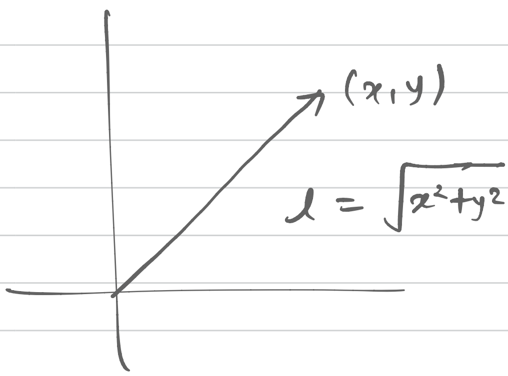
$$\Rightarrow \begin{bmatrix} -2k+k & k \\ k & -2k+k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -k & k \\ k & -k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -kx_1 + kx_2 &= 0 \\ kx_1 - kx_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 = x_2$$

look for X s.t.

$$\sqrt{x_1^2 + x_2^2} = 1$$



$$\Rightarrow x_1^2 + x_2^2 = 1$$

$$\Rightarrow 2x_1^2 = 1 \quad \Rightarrow \quad \boxed{x_1 = \frac{1}{\sqrt{2}}} \quad \boxed{x_2 = \frac{1}{\sqrt{2}}}$$

$\lambda = -k,$ $\quad X = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \rightarrow X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenvectors are not unique

↳ if X is a eigenvector corresponding to eigenvalue λ of matrix A

$$AX = \lambda X$$

then $Y = \alpha X$ is also eigenvector.

$$\downarrow$$

$$AY = \lambda Y$$

$$\begin{aligned} AY = A(\alpha X) &= \alpha AX \\ &= \alpha \lambda X \\ &= \lambda(\alpha X) \\ &= \lambda Y \end{aligned}$$

$$\Rightarrow \boxed{AY = \lambda Y}$$

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$\lambda = -3k$, $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -3k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2k+3k & k \\ k & -2k+3k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left. \begin{aligned} k x_1 + k x_2 &= 0 \\ k x_1 + k x_2 &= 0 \end{aligned} \right\} \rightarrow \boxed{x_1 = -x_2}$$

<u>additional equation</u>

$$\sqrt{x_1^2 + x_2^2} = 1 \Rightarrow x_1^2 + x_2^2 = 1$$

$$\Rightarrow x_1 = \frac{1}{\sqrt{2}}$$

$$x_2 = -\frac{1}{\sqrt{2}}$$

for $\lambda = -3k$, $X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

<u>ODE</u>

1. $\boxed{\dfrac{d^2u}{dt^2} = Au, \quad u(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}}$

<u>IC</u>

$$u(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{du}{dt}(0) = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \begin{array}{l} \rightarrow v_0 \\ w_0 \end{array}$$

To build incomplete / characteristics solution of ①,

$u(t) = \sin(\omega t) X$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\boxed{\omega \,\&\, X \text{ are constants}}$

for any ω, X $\quad \dfrac{d^2u}{dt^2} = -\omega^2 u$

but we want $\quad \dfrac{d^2u}{dt^2} = Au$

find ω and vector X such that

$$Au = -\omega^2 u$$

new notation $\quad \lambda = -\omega^2$

$$Au = \lambda u$$

$$\Rightarrow A(\sin(\omega t) X) = \lambda \sin(\omega t) X$$

$$\Rightarrow \cancel{\sin}(\omega t)(AX) = \cancel{\sin}(\omega t)(\lambda X)$$

$$\Rightarrow \boxed{AX = \lambda X}$$

where λ and X are unknown

$\Downarrow$

eigenvalue - eigenvector problem

λ = eigenvalue

X = eigenvector

1st pair $\lambda_1 = -k$, $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\downarrow$

$\lambda_1 = -\omega_1^2 = -k$

$\Rightarrow \quad \omega_1 = \sqrt{k}$

$$u_1(t) = \sin(\omega_1 t)\, X_1 \quad \rightarrow \quad \frac{d^2 u_1}{dt^2} = A u_1$$

2nd pair

$\lambda_2 = -3k$, $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\downarrow$

$\omega_2 = \sqrt{3k}$

$$u_2(t) = \sin(\omega_2 t)\, X_2 \quad \longrightarrow \quad \frac{d^2 u_2}{dt^2} = A u_2$$

Linear combination of eigenvectors

Take α_1, α_2 any two numbers

and let

$$f = f(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$$

$$\frac{d^2 f}{dt^2} = \alpha_1 \frac{d^2 u_1}{dt^2} + \alpha_2 \frac{d^2 u_2}{dt^2}$$

$$= \alpha_1 A u_1 + \alpha_2 A u_2$$

$$= A(\alpha_1 u_1) + A(\alpha_2 u_2)$$

$$\left(A(a+b) = Aa + Ab \right)$$

$$= A(\alpha_1 u_1 + \alpha_2 u_2)$$

$$= Af$$

$$\Rightarrow \quad \boxed{\frac{d^2 f}{dt^2} = Af} \quad \text{for any } \alpha_1, \alpha_2$$

find α_1, α_2 such that $f = \alpha_1 u_1 + \alpha_2 u_2$,

and

$$f(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \text{trivially true}$$

$$\frac{df}{dt}(0) = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

$\downarrow$

$$\frac{df}{dt} = \alpha_1 \frac{du_1}{dt} + \alpha_2 \frac{du_2}{dt}$$

$$= \omega_1 \alpha_1 \cos(\omega_1 t) X_1 + \omega_2 \alpha_2 \cos(\omega_2 t) X_2$$

$$\frac{df}{dt}(0) = \omega_1 \alpha_1 X_1 + \omega_2 \alpha_2 X_2 = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \omega_1 \alpha_1 + \omega_2 \alpha_2 \\ -\omega_1 \alpha_1 + \omega_2 \alpha_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} \omega_1 \alpha_1 + \omega_2 \alpha_2 &= a_0 \\ -\omega_1 \alpha_1 + \omega_2 \alpha_2 &= b_0 \end{aligned}}$$

system of linear equation for α_1, α_2.

$\downarrow$

find solution α_1, α_2

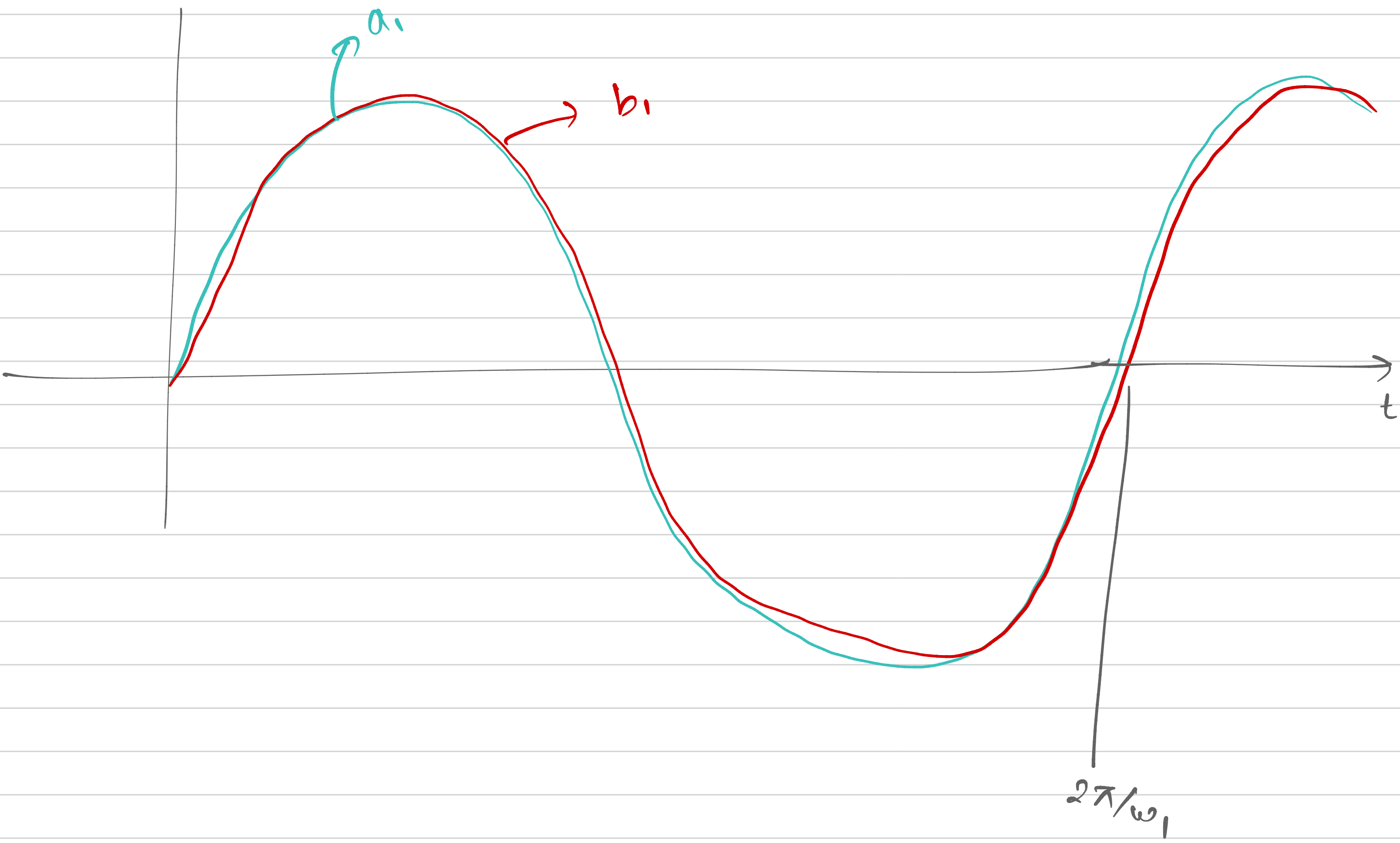
$$f = \alpha_1 u_1 + \alpha_2 u_2 \longrightarrow$$

① $\dfrac{d^2 f}{dt^2} = Af$

② $f(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \dfrac{df}{dt}(0) = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = u_1 = \sin(\omega_1 t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \omega_1 = \sqrt{K}$$

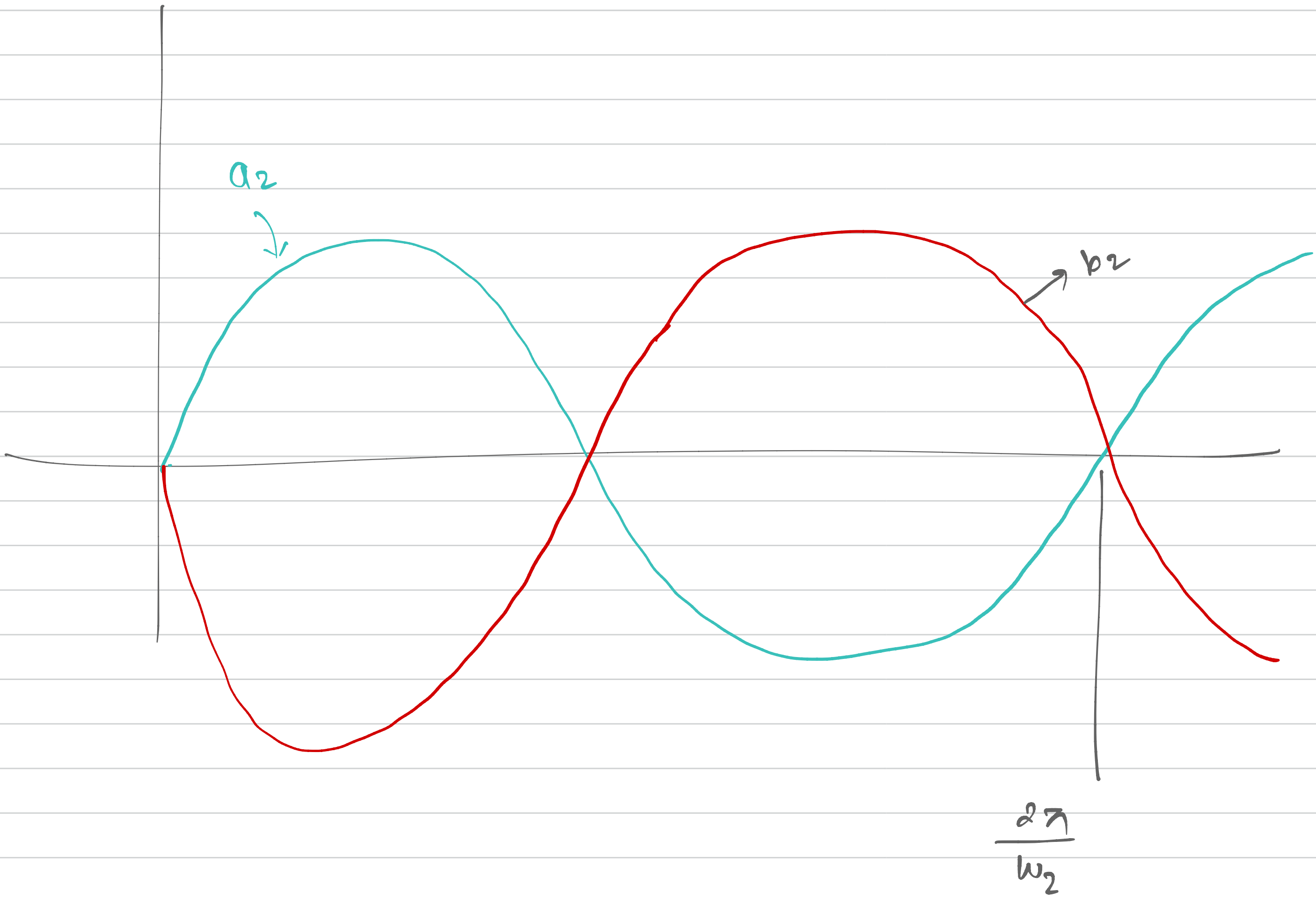
$$\left.\begin{aligned} a_1(t) &= \sin(\omega_1 t) \\ b_1(t) &= \sin(\omega_1 t) \end{aligned}\right\} \Rightarrow a_1(t) = b_1(t)$$



$$u_2(t) = \sin(\omega_2 t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

$$a_2 = \sin(\omega_2 t)$$

$$b_2 = -\sin(\omega_2 t)$$



a_2
b_2
$\frac{2\pi}{w_2}$