Lecture 17

Topic 4
- Linear system of equations using matrix and vector
- Studied a bit about matrix
- Methods to solve $Ax = b$
- ✓ LU factorization
- ✓ Iterative (numerical) methods for $Ax = b$
- Inverse of A^{-1}, condition number of a matrix A

Topic 5
- Eigenvalues and eigenvectors of a matrix A

Types of matrix

- Square matrix # rows = # columns
- Symmetric matrix A square matrix A is symmetric if $A^T = A \Rightarrow a_{ij} = a_{ji}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \longrightarrow A^T = A$$

- Skew-symmetric (anti-symmetric) matrix:

$$A^T = -A \implies a_{ij} = -a_{ji}$$

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} \Rightarrow \qquad A^T = -A$$

<u>Any</u> square matrix A can be <u>uniquely</u> written as sum of symmetric and skew-symmetric matrices

$$A = A_{sym} + A_{skew}$$

$$A_{sym} = \frac{1}{2}(A + A^T) \qquad\qquad A_{skew} = \frac{1}{2}(A - A^T)$$

- <u>Diagonal matrix</u>

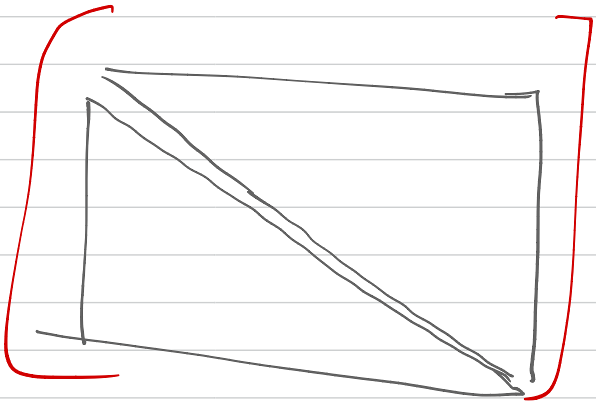
  A is diagonal matrix if $a_{ij} = 0$ for <u>any</u> $i \neq j$

$$\begin{bmatrix} \boxed{\begin{matrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{matrix}} \end{bmatrix}$$

- Triangular matrix

  - Upper triangle matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}$$



for all $j = 1, 2, \ldots, n$

$a_{ij} = 0$ for any $i > j$

  - Lower triangle matrix

for all $j = 1, 2, \ldots, n$

$a_{ij} = 0$ for any $i < j$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & - & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$\longrightarrow A = A_{sym} + A_{skew}$

$\longrightarrow A = A_{upper} + A_{lower} + A_{diag}$

$A_{diag} + A_{upper} \rightarrow$ upper triangle

$$A_{diag} = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$$

- LU factorization of a matrix A

Given $A_{n \times n}$, lower triangle matrix $L_{n \times n}$, and upper triangle matrix $U_{n \times n}$.

Then L and U are called LU factorization matrice of A if

$$A = L\,U$$

- LU factorization is possible for invertible matrix A

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\downarrow$ 1st step of foward elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} \end{bmatrix}$$

$$\rightarrow f_{21} = \frac{a_{21}}{a_{11}}$$

$$f_{31} = \frac{a_{31}}{a_{11}}$$

$$a_{22}^{(1)} = a_{22} - f_{21}\, a_{12}$$

$$a_{23}^{(1)} = a_{23} - f_{21}\, a_{13}$$

$$b_2^{(1)} = b_2 - f_{21}\, b_1$$

$$a_{32}^{(1)} = a_{32} - f_{31}\, a_{12}$$

$$a_{33}^{(1)} = a_{33} - f_{31}\, a_{13}$$

$$b_3^{(1)} = b_3 - f_{31}\, b_1$$

$\downarrow$ 2nd step

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \end{bmatrix}$$

$$f_{32} = \frac{a_{32}^{(1)}}{a_{22}^{(1)}}$$

$$a_{33}^{(2)} = a_{33}^{(1)} - f_{32}\, a_{23}^{(1)}$$

$$b_3^{(2)} = b_3^{(1)} - f_{32}\, b_2^{(1)}$$

Define

- $$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}$$

- $$L = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}$$

$$LU = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \cancel{f_{21} a_{11}} \; a_{21} & \cancel{f_{21} a_{12} + a_{22}^{(1)}} \; a_{22} & a_{23} \\ a_{31} & a_{32} & \cancel{f_{31} a_{13} + f_{32} a_{23}^{(1)} + a_{33}^{(2)}} \\ & & a_{33} \end{bmatrix}$$

- $f_{21} a_{11} = \frac{a_{21}}{a_{11}} a_{11} = a_{21}$

- $f_{21} a_{12} + a_{22}^{(1)} = \cancel{f_{21} a_{12}} + a_{22} - \cancel{f_{21} a_{12}}$

$$= a_{22}$$

- $f_{31} a_{13} + f_{32} a_{23}^{(1)} + a_{33}^{(2)}$

$$= f_{31} a_{13} + \cancel{f_{32} a_{23}^{(1)}} + a_{33}^{(1)} - \cancel{f_{32} a_{23}^{(1)}}$$

$$= \cancel{f_{31} a_{13}} + a_{33} - \cancel{f_{31} a_{13}}$$

$$= a_{33}$$

Solve $Ax = b$ using LU facturization

$$A = LU$$

$$\boxed{Ax = b}$$

$$\Rightarrow LUx = b$$

$$\Rightarrow L(Ux) = b \implies$$

- $d = Ux$
- $Ld = b$

- Solve for d using $Ld = b$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ \vdots & & \vdots \\ l_{n1} & l_{n2} & l_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

(1) $l_{11} d_1 = b_1$

(2) $l_{21} d_1 + l_{22} d_2 = b_2$

$\vdots$

(n) $l_{n1} d_1 + \dots + l_{nn} d_n = b_n$

$$d_1 = \frac{b_1}{l_{11}}$$

$$d_2 = \frac{b_2 - l_{21} d_1}{l_{22}}$$

$$d_3 = \frac{b_3 - l_{31} d_1 - l_{32} d_2}{l_{33}}$$

$\vdots$

$$d_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} d_j}{l_{ii}}$$

- Solve for x using $Ux = d$

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ 0 & \vdots & & \\ \vdots & & & \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

1. $u_{11} x_1 + \cdots + u_{1n} x_n = d_1$
2. $u_{22} x_2 \cdots + u_{2n} x_n = d_2$

$\vdots$

(n-1) $u_{(n-1)(n-1)} x_{n-1} + u_{(n-1)n} x_n = d_{n-1}$

(n) $u_{nn} x_n = d_n$

- $$x_n = \frac{d_n}{u_{nn}}$$

- $$x_{n-1} = \frac{d_{n-1} - u_{(n-1)n} x_n}{u_{(n-1)(n-1)}}$$

$\vdots$

- $$x_i = \frac{d_i - \sum_{j=i+1}^{n} u_{ij} x_j}{u_{ii}}$$

- Cholesky factorization A symmetric matrix A can be written as

$$A = U^T U$$

where

U is upper triangle matrix