Lecture 14

Methods to solve $Ax = b$

(i) Graphical method

(ii) Direct method

(A.) Inverse of matrix $\quad x = A^{-1}b$

(B.) Cramer's rule

(iii) Numerical method

(A.) Gauss-elimination

(B.) An improved Gauss-elimination

Cramer's Rule

$\underbrace{\text{2 equations}}_{\#\text{ rows in } A}$, $\underbrace{\text{2 unknowns}}_{\#\text{ columns in } A}$

① $a_{11}x_1 + a_{12}x_2 = b_1$

② $a_{21}x_1 + a_{22}x_2 = b_2$

② $- \dfrac{a_{21}}{a_{11}}$ ①

$$\left(a_{21} - \frac{a_{21}}{a_{11}}a_{11}\right)x_1 + \left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 = b_2 - \frac{a_{21}}{a_{11}}b_1$$

$$\Rightarrow \left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 = b_2 - \frac{a_{21}}{a_{11}}b_1$$

$$\Rightarrow (a_{11}a_{22} - a_{21}a_{12})x_2 = a_{11}b_2 - a_{21}b_1$$

$$\Rightarrow \boxed{x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}}$$

Substitute x_2 in equation ①

$$a_{11} x_1 + a_{12} \left(\frac{a_{11} b_2 - a_{21} b_1}{a_{11} a_{22} - a_{21} a_{12}} \right) = b_1$$

$$\Rightarrow \quad x_1 = \frac{1}{a_{11}} \left[b_1 - \frac{(a_{12} a_{11} b_2 - a_{12} a_{21} b_1)}{a_{11} a_{22} - a_{21} a_{12}} \right]$$

$$= \frac{1}{a_{11}} \left[\frac{b_1 a_{11} a_{22} - b_1 \cancel{a_{21}} a_{12} - a_{12} a_{11} b_2 + a_{12} \cancel{a_{21}} b_1}{a_{11} a_{22} - a_{21} a_{12}} \right]$$

$$\boxed{x_1 = \frac{a_{22} b_1 - a_{12} b_2}{a_{11} a_{22} - a_{21} a_{12}}} = \frac{D_1}{D}$$

$$\boxed{x_2 = \frac{a_{11} b_2 - a_{21} b_1}{a_{11} a_{22} - a_{21} a_{12}}} = \frac{D_2}{D}$$

- $\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11} a_{22} - a_{21} a_{12} = D$
- $\det\left(\begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}\right) = b_1 a_{22} - b_2 a_{12} = D_1$
- $\det\left(\begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}\right) = a_{11} b_2 - a_{21} b_1 = D_2$

3 equations, 3 unknowns

① $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

② $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

③ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

step 1 : $② - \frac{a_{21}}{a_{11}}①$, $③ - \frac{a_{31}}{a_{11}}①$

④ $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

⑤ $0 + a'_{22}x_2 + a'_{23}x_3 = b'_2$

⑥ $0 + a'_{32}x_2 + a'_{33}x_3 = b'_3$

step 2 $⑥ - \frac{a'_{32}}{a'_{22}}⑤$

⑦ $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

⑧ $0 + a'_{22}x_2 + a'_{23}x_3 = b'_2$

⑨ $0 + 0 + a''_{33}x_3 = b''_3$

✓ $x_3 = \frac{b''_3}{a''_{33}}$

✓ $x_2 = \frac{b'_2 - a'_{23}x_3}{a'_{22}}$

✓ $x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$

$$x_1 = \frac{\det\left(\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{23} \end{bmatrix}\right)}{\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{bmatrix}\right)} = \frac{D_1}{D}$$

$$x_2 = \frac{\det\left(\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}\right)}{\det\left(\begin{bmatrix} a_{11} & \cdot & \cdot \\ a_{21} & - & \cdot \\ a_{31} & - & \end{bmatrix}\right)} = \frac{D_2}{D}$$

$$x_3 = \frac{\det\left(\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}\right)}{\det(\,\cdot\cdot\,)} = \frac{D_3}{D}$$

General n equations and n unknowns system

for $i = 1, 2, \ldots, n$

$$x_i = \frac{D_i}{D} = \frac{\det\left(\begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & & b_2 & & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & & b_n & & a_{nn} \end{bmatrix}\right)}{\det(A)}$$

Compute total $n+1$ determinants of $n+1$ matrices

Numerical method : Gauss- Elimination method

$\boxed{\begin{array}{ll} \text{①} & \text{Forward elimination} \\ \text{②} & \text{Backward substitution} \end{array}}$

Forward elimination

① $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

② $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

$\vdots$

ⓝ $a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$

A

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

step 1 : ② $- \dfrac{a_{21}}{a_{11}}$ ① , ③ $- \dfrac{a_{31}}{a_{11}}$ ① , $\ldots$, ⓝ $- \dfrac{a_{n1}}{a_{11}}$ ①

① $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

② $0x_1 + a_{22}^{(1)}x_2 + \cdots + a_{2n}^{(1)}x_n = b_2^{(1)}$

③ $0x_1 + a_{32}^{(1)}x_2 + \cdots + a_{3n}^{(1)}x_n = b_3^{(1)}$

$\vdots$

ⓝ $0x_1 + a_{n2}^{(1)}x_2 + \cdots + a_{nn}^{(1)}x_n = b_n^{(1)}$

$\Rightarrow$

$$a_{22} \rightarrow a_{22}^{(1)} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}$$

$$a_{2n} \rightarrow a_{2n}^{(1)} = a_{2n} - \frac{a_{21}a_{1n}}{a_{11}}$$

step 2 : ③ $- \dfrac{a_{32}^{(1)}}{a_{22}^{(1)}}$ ② , ④ $- \dfrac{a_{42}^{(1)}}{a_{22}^{(1)}}$ ② , $\ldots$, ⓝ $- \dfrac{a_{n2}^{(1)}}{a_{22}^{(1)}}$ ②

(1) $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

(2) $0x_1 + a_{22}^{(1)}x_2 + \cdots + a_{2n}^{(1)}x_n = b_2^{(1)}$

(3) $0x_1 + 0x_2 + a_{33}^{(2)}x_3 + \cdots + a_{3n}^{(2)}x_n = b_3^{(2)}$

$a_{33} \longrightarrow a_{33}^{(2)}$

$\vdots$

(n) $0x_1 + 0x_2 + a_{n3}^{(2)}x_3 + \cdots + a_{nn}^{(2)}x_n = b_n^{(2)}$

$$(4) - \frac{a_{43}^{(2)}}{a_{33}^{(2)}}(3), \quad \ldots, \quad (n) - \frac{a_{n3}^{(2)}}{a_{33}^{(2)}}(3)$$

step n-1:

(1) $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

(2) $0x_1 + a_{22}^{(1)}x_2 + \cdots + a_{2n}^{(1)}x_n = b_2^{(2)}$

(3) $0x_1 + 0x_2 + a_{33}^{(2)}x_3 + \cdots + a_{3n}^{(2)}x_n = b_3^{(2)}$

$\vdots$

(n-1) $0x_1 + 0x_2 + \cdots + 0x_{n-2} + a_{(n-1)(n-1)}^{(n-2)}x_{n-1} + a_{(n-1)n}^{(n-2)}x_n = b_{n-1}^{(n-2)}$

(n) $0x_1 + 0x_2 + 0x_3 + \cdots + 0x_{n-1} + a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$

Backward substitution

(1) $$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

(2) $$x_{n-1} = \frac{b_{n-1}^{(n-2)} - a_{(n-1)n}^{(n-2)}x_n}{a_{(n-1)(n-1)}^{(n-2)}}$$

$\vdots$

(i) $$x_i = \frac{\left[b_i^{(i-1)} - a_{i(i+1)}^{(i-2)} x_{i+1} - a_{i(i+2)}^{(i-2)} x_{i+2} - \cdots - a_{in}^{(i-2)} x_n \right]}{a_{ii}^{(i-1)}}$$

$$\boxed{x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}}$$

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

for $i = (n-1) : -1 : 1$

$c = 0$

for $j = (i+1) : n$

$c = c + a_{ij}^{(i-1)} x_j$

end

$$x_i = \frac{b_i^{(i-1)} - c}{a_{ii}^{(i-1)}}$$

end