Lecture 12

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

i^{th} row , j^{th} column $\longrightarrow$ a_{ij}

- addition $A_{m \times n}$ $B_{m \times n}$ (element-wise)

$$C = A + B \Rightarrow [c_{ij}] = [a_{ij} + b_{ij}]$$

- multiplication by scalar α , $A_{m \times n}$

$$C_{m \times n} = \alpha A_{m \times n} \Rightarrow [c_{ij}] = [\alpha a_{ij}] \quad \text{(element-wise)}$$

- Subtraction

$$C = A - B \Rightarrow [c_{ij}] = [a_{ij} - b_{ij}]$$

- Multiplication of matrices $A_{m \times n}$, $B_{l \times r}$

(i) $C = A \times B$ $\longrightarrow$ defined only if

\# columns of A = \# rows of B

$\longrightarrow$ size(C) : $m \times r$

$$[c_{ij}] = \left[\sum_{k=1}^{n} a_{ik} \, b_{kj} \right]$$

(ii) $D = B \times A$ # columns of B = # rows of A

size $(D) = l \times n$

$$[d_{ij}] = \left[\sum_{k=1}^{m} b_{ik}\, a_{kj}\right]$$

multiplication of matrix & vector

(i) treat row vector as matrix $1 \times n$ $[\, \cdot \ \cdot \ \cdot \ \cdot \,]$

treat column vector as matrix $n \times 1$ $\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$

$A_{m \times n}$, $n > 1$

$A_{m \times n}\, x$ only if # of rows of $x = n$

$\downarrow$

x has to be column vector

$$y_{m \times 1} = A_{m \times n}\, x_{n \times 1} \Rightarrow y_{ij} = \sum_{k=1}^{n} a_{ik}\, x_{kj}$$

since x is a column vector,
I can write x_{kj} as simply x_k

y is also column vector, I can $y_{ij} \rightarrow y_i$

$$\Rightarrow \boxed{y_i = \sum_{k=1}^{n} a_{ik}\, x_k}$$

$F_1, \quad F_2, \quad F_3, \quad R_{2y}, \quad R_{3x}, \quad R_{3y}$

$\downarrow \quad \downarrow \quad\quad\quad\quad\quad \downarrow$

$x_1 \quad x_2 \quad\quad\quad\quad\quad x_6$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_6 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{16} \\ a_{21} & a_{22} & \cdots & a_{26} \\ \vdots & & & \\ a_{61} & & \cdots & a_{66} \end{bmatrix}$$

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{16}x_6 = b_1 \\ \vdots \\ a_{61}x_1 + a_{62}x_2 + \cdots + a_{66}x_6 = b_6 \end{matrix}\right\} \Rightarrow \boxed{Ax = b}$$

Solve x such that $Ax = b$

$A_{m \times n}$

- Methods to solve $Ax = b$

(i) Graphical method (limited & works for system say 2 or 3 equations)

(ii) Direct method
- Inverse of matrix
- Cramer's rule

} suitable only for system with say 2 to 6 equations

(iii) Numerical method
- Iterative method (exact)
- Partial-pivoted iterated method

(i) We will consider only $A_{n \times n}$, n any integer, $\Leftarrow$ Square

$\hookrightarrow$ n equations and n unknowns

(i) $m > n \longrightarrow$ over determined system

(ii) $m < n \longrightarrow$ under determined system

- **Graphical method.**

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

① — $a_{11} x_1 + a_{12} x_2 = b_1$

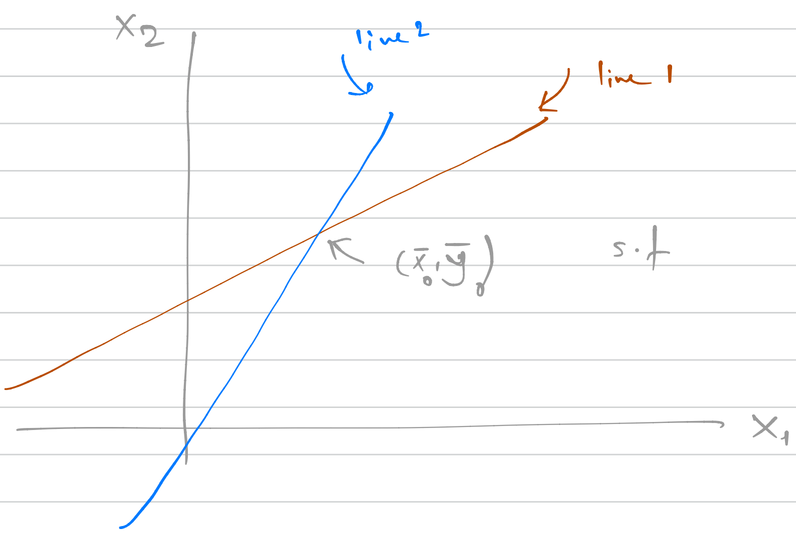
② — $a_{21} x_1 + a_{22} x_2 = b_2$

define $\bar{x}$ variable $\longrightarrow x_1$

$\bar{y}$ variable $\longrightarrow x_2$

$$a_{11} \bar{x} + a_{12} \bar{y} = b_1$$

$$\Rightarrow \boxed{\bar{y} = \frac{b_1}{a_{12}} - \frac{a_{11}}{a_{12}} \bar{x}}$$



s.t

$$\bar{y}_0 = \frac{b_1}{a_{12}} - \frac{a_{11}}{a_{12}} \bar{x}_0 \leftarrow \text{line 1}$$

$$\bar{y}_0 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} \bar{x}_0 \leftarrow \text{line 2}$$

$$\boxed{Ax = b}$$

multiply both sides of above equation with A^{-1}

$\Rightarrow A^{-1}(Ax) = A^{-1}(b)$

$\Rightarrow \boxed{x = A^{-1} b}$

I know another matrix B such that

$$A(Bx) = x$$

$$B(Ax) = x$$

then B is inverse of A and you write B as A^{-1}

$\Downarrow$

inverse is defined only for square matrix

$A_{m \times n}$, $B_{r \times s}$ $\Rightarrow m = n$

$\Rightarrow r = m = s = n$

Direct method: Inverse of a matrix

$Ax = b$, A^{-1}

$$\Rightarrow \boxed{x = A^{-1} b}$$

- $n = 1$: $A = [a_{11}]$, $A^{-1} = \left[\frac{1}{a_{11}}\right]$, $\det(A) = a_{11}$

- $n = 2$: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A^{-1} = \frac{1}{(\det A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{21} a_{12}$$

- $n = 3$: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{12} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{bmatrix}$

$$\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- $n > 3$

- **Cramer's Rule** 2 equations & 2 unknowns

(1) $a_{11}x_1 + a_{12}x_2 = b_1$

(2) $a_{21}x_1 + a_{22}x_2 = b_2$

$a_{21} \times (1) \Rightarrow a_{21}a_{11}x_1 + a_{21}a_{12}x_2 = a_{21}b_1$ — (3)

$a_{11} \times (2) \Rightarrow a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = a_{11}b_2$ — (4)

(3) − (4)

$$(a_{21}a_{12} - a_{11}a_{22})x_2 = a_{21}b_1 - a_{11}b_2$$

$$\Rightarrow \boxed{x_2 = \frac{a_{21}b_1 - a_{11}b_2}{a_{21}a_{12} - a_{11}a_{22}}}$$