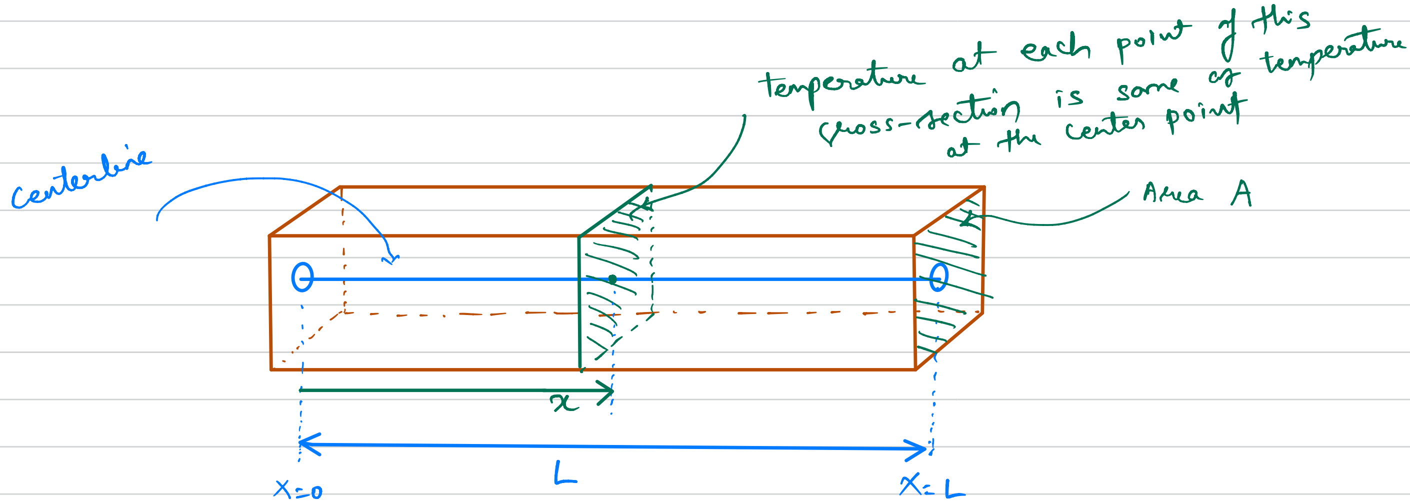
Temperature in a one-dimensional metal bar



— Consider a bar of uniform cross-section A and of length L. We are interested in modeling temperature at every points inside the bar.

— We assume that temperature at every point in the cross-section is some and equal to the temperature at the center of cros-section.

— This assumption allows us to consider a temperature (say in degree celcius)

$$T = T(x) \quad , \quad 0 \leq x \leq L$$

as a function of point on the centerline inside the bar, see figure. Here x, between 0 and L, is parametric coordinate of a point on the centerline of the bar.

— Thus, we have temperature T as a function of one variable x.

— We are observing bar in time interval $[0, t_F]$ and we assume that t_F is small such that temperature is constant in time. I.e. T just depends on distance of point on centerline from $x = 0$.

External conditions :

1. We assume that temperature at left and right end of the bar is fixed to prescribed values :

$$T(x=0) = T(0) = T_0$$

$$T(x=L) = T(L) = T_L$$

where T_0 and T_L are numbers.

2. Bar is placed in a surrounding such that at every point x along the center line, bar is supplied with external heat energy.

We let

$q_{ext}(x) =$ external heat per unit volume per unit time at x

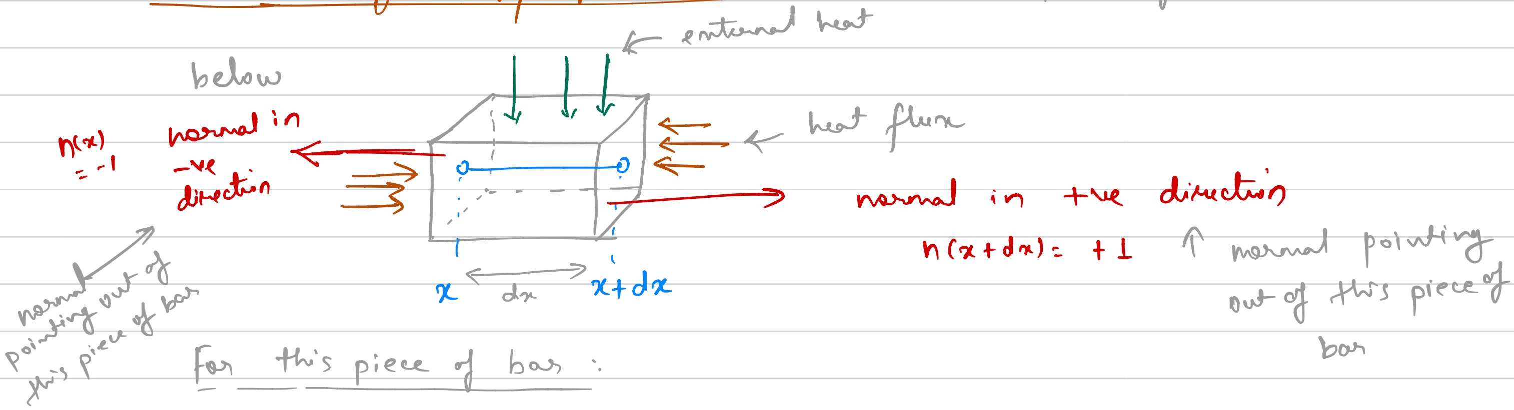
This is a known function of x

units of q_{ext}

$= \dfrac{\text{energy}}{\text{time} \times \text{volume}} = \dfrac{\text{Watt}}{m^3}$

Conservation of energy principle

Consider a part of the bar as shown below



For this piece of bar :

Rate of energy into it from x end + Rate of energy into it from $(x+dx)$ end

+ Rate of external energy into from x to $x+dx$ length

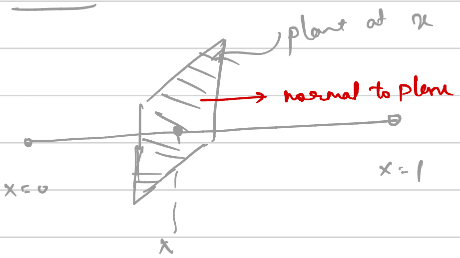
⟹ Heat flux : Using Fourier's law, the rate of heat flux at cross-section passing through centerline point x is

$$q_{flux}(x) = -K \frac{dT(x)}{dx}$$ [I.e. heat flux is proportional to gradient of temperature]

where K is the positive number and generally depends on the material of the bar.

units of q are $\frac{\text{energy}}{\text{time} \times \text{temperature} \times \text{length}} = \frac{\text{Watt}}{\text{m} \times \text{celcius}}$

Note : At point X, consider a plane



Flux is defined at this plane and

(i) flux in right direction $= q_{flux}(x) \times n(x) = q_{flux}(x) \times 1$ (n: normal)

(ii) flux in left direction $= q_{flux}(x) \times n(x) = -q_{flux}(x)$

⟹ Rate of total external heating of bar of length dx

$= (\text{rate of external heat per unit volume at point } x) \times (\text{length}) \times (\text{Area})$

$\approx q_{ext}(x)\, dx\, A$ where q_{ext} is given function

⟹ From ⊛,

$$A\, q_{flux}(x)\, n(x) + A\, q_{flux}(x+dx)\, n(x+dx) = q_{ext}(x)\, dx\, A$$

(i) A is cross-section area
(ii) $n(x) = -1$
$n(x+dx) = 1$

$$\Rightarrow A\, q_{flux}(x)(-1) + A\, q_{flux}(x+dx)(1) = A\, q_{ext}(x)\, dx$$

$$\Rightarrow A\, q_{flux}(x+dx) - A\, q_{flux}(x) = A\, q_{ext}(x)\, dx$$

$$\Rightarrow A\left[\frac{q_{flux}(x+dx) - q_{flux}(x)}{dx}\right] = A\, q_{ext}(x)$$

Thus, if dx is small, we have

$$\frac{q_{flux}(x+dx) - q_{flux}(x)}{dx} \approx \frac{d}{dx} q_{flux}(x)$$

and therefore

$$\boxed{A \frac{d\, q_{flux}(x)}{dx} = A\, q_{ent}(x)}$$

where $q_{flux}(x) = -k \frac{dT(x)}{dx}$

Substituting $q_{flux}(x) = -k \frac{dT(x)}{dx}$

$$A \frac{d}{dx}\left(-k \frac{dT(x)}{dx}\right) = A\, q_{ent}(x)$$

$$\Rightarrow \boxed{-Ak \frac{d^2T(x)}{dx^2} = A\, q_{ent}(x)} \qquad (1)$$

where we used the fact that k is constant and thus

$$\frac{d}{dx}\left(-k \frac{dT}{dx}\right) = -k \frac{d^2T}{dx^2} - \cancel{\frac{dk}{dx}}^{\,0} \frac{dT}{dx} = -k \frac{d^2T}{dx^2}$$

— Thus, temperature T satisfies second order ordinary differential equation. To solve (1) completely we either need two initial condition or two boundary condition.

Final equation of temperature in the bar

$$\boxed{-kA\frac{d^2T(x)}{dx^2} = A\,q_{ext}(x) \quad \text{for} \quad 0 \le x \le L} \quad \text{— (2)}$$

$$\boxed{\begin{aligned} T(0) &= T_0 \\ T(L) &= T_L \end{aligned}} \Rightarrow \text{Boundary conditions}$$

— The above set of equations are called Boundary value Problem — Ordinary differential equation (BVP-ODE).

— $q_{ext}(x)$ is a known function and A, k, T_0, T_L are given numbers.

Analytical solution : let

$L = 1$ meter

$A = 1$ (meter)^2

$T_0 = 0$ degree celcius

$T_L = 100$ degree celcius

$k = \frac{1}{200} \frac{\text{Watt}}{\text{Celcius} \times \text{meter}}$

Further, let

$$q_{ext}(x) = 12x^2 + 50\cos(5x) + 100\,x\sin(10x)$$

from (2) :

$$-kA\frac{d^2T}{dx^2} = A\,q_{ext}(x)$$

$$\Rightarrow \frac{d^2 T(x)}{dx^2} = -\frac{q_{ent}(x)}{k}$$

$$\Rightarrow \int_0^y \frac{d^2 T(x)}{dx^2} dx = -\frac{1}{k}\int_0^y q_{ent}(x)\, dx$$

$$\Rightarrow \frac{dT(y)}{dx} - \frac{dT(0)}{dx} = -\frac{1}{k}\int_0^y q_{ent}(x)\, dx$$

let us call it a ← number that we will determine soon.

let $Q_1(y) := \int_0^y q_{ent}(x)\, dx$

$Q_2(y) := \int_0^y Q_1(x)\, dx$

$$\Rightarrow \frac{dT(y)}{dx} = a - \frac{1}{k} Q_1(y)$$

integrating from $y=0$ to $y=z$

$$T(z) - T(0) = az - \frac{1}{k} Q_2(z)$$

$$\Rightarrow T(z) = T(0) + az - \frac{1}{k} Q_2(z)$$

$\because T(0) = T_0$ ← given

$$\Rightarrow \boxed{T(z) = T_0 + az - \frac{1}{k} Q_2(z)}$$

To determine a: use $T(L) = T_L$ ← given

$$\Rightarrow T_0 + aL - \frac{1}{k} Q_2(L) = T_L$$

$$a = \frac{\left(T_L - T_0 + \frac{1}{K} Q_2(L)\right)}{L}$$

Thus

$$\boxed{T(z) = T_0 + \frac{\left(T_L - T_0 + \frac{1}{K} Q_2(L)\right)}{L} z - \frac{1}{K} Q_2(z)}$$

where $L = 1$, T_0, T_L, K given numbers.

Exact solution where

$Q_1(x) = \int_0^x q_{ext}(y)\,dy$

$Q_2(x) = \int_0^x Q_1(y)\,dy$

for $q_{ext}(x) = 12x^2 + \cos(5x) + 100x\sin(10x)$

Using Matlab symbolic library

$$Q_1(x) = \int_0^x q_{ext}(y)\,dy = \frac{\sin(5x)}{5} + \sin(10x) + 10x\left(2(\sin(5x))^2 - 1\right) + 4x^3$$

$$Q_2(x) = \int_0^x Q_1(y)\,dy = x^4 - x\sin(10x) - \frac{2(\cos(5x))^2}{5} - \frac{\cos(5x)}{25} + \frac{11}{25}$$