Assignment 2

In this assignment, we will solve for the temperature in a metal bar which is fixed at temperature $T_0 = 0$ degrees (Celsius) at the left end and $T_L = 100$ degrees (Celsius) at the right end.

We can assume that temperature T is only a function of x-coordinate along the bar, i.e., $T = T(x)$, and with this and the other simplifying assumptions, we get the following second-order Ordinary Differential Equation for T:

$$-\kappa A \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} = A q_{ext}(x), \qquad \text{for all } 0 \leq x \leq L. \tag{1}$$

The boundary conditions are

$$T(x = 0) = T(0) = T_0, \qquad T(x = L) = T(L) = T_L. \tag{2}$$

Here, A is the area of the cross-section of the bar (we assume that the shape and size of the cross-section is same throughout the length, see fig. 1), $\kappa > 0$ a positive constant (from Fourier's law) referred to as heat conductivity, $q_{ext} = q_{ext}(x)$ external heat supplied at x, L is the length of the bar.

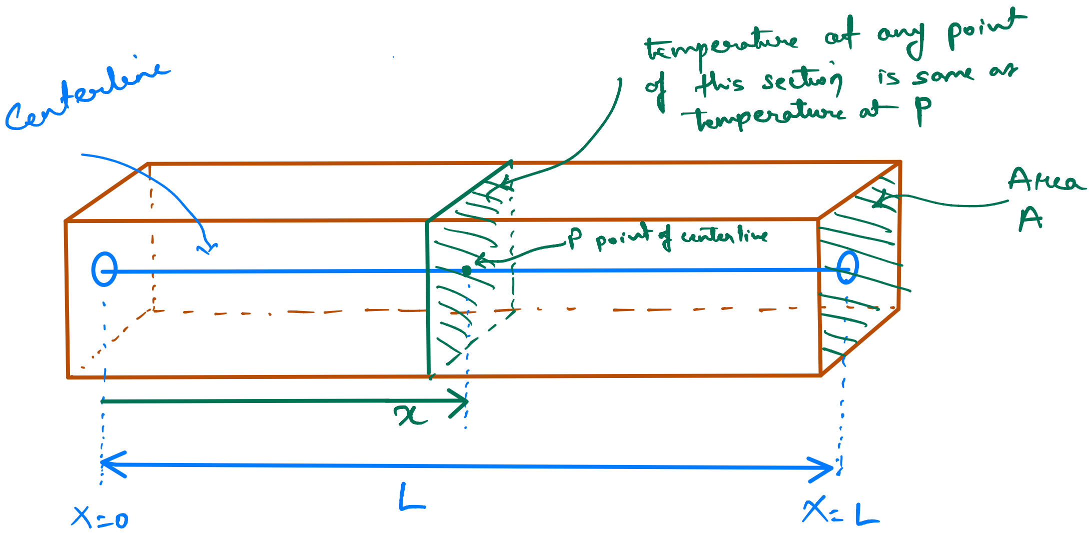

Figure 1: Thermal heating of the bar with a uniform cross-section.

Remark 1. See the supplementary file 'A2_heating_bar.pdf' for the derivation.

Parameters. Let $L = 1$ m, $A = 1$ m^2, $T_0 = 0$ Celsius, $T_L = 100$ Celsius, and $\kappa = 1/200$ Watts/(m$\times$Celsius). Further, fix external heat function (in units of Watt/(m^3) as follows:

$$q_{ext}(x) = 12x^2 + \cos(5x) + 100x\sin(10x). \tag{3}$$

Remark 2. Read carefully the problems. They all are easy and results can be verified by just plotting the temperature in matlab and observing it. I have also included the hints in the file 'A2_hints.pdf'. Before you panic (I hope not!!), do check the hints; some problems are practically solved in the hints file.

Problem 1 (10 marks). *Exact solution* of (1) with boundary conditions (2) is given by

$$T(x) = T_0 + \left[\frac{T_L - T_0 + \frac{1}{\kappa}Q_2(L)}{L}\right] x - \frac{1}{\kappa}Q_2(x), \qquad \text{for } x \in [0, L], \tag{4}$$

where $Q_2 = Q_2(x)$ is a function of x and is given by

$$Q_2(x) = \int_0^x Q_1(y)\mathrm{d}y. \tag{5}$$

And, finally Q_1 function is defined as

$$Q_1(x) = \int_0^x q_{ext}(y)\mathrm{d}y. \tag{6}$$

(i) For q_{ext} given in (3) and the parameter values specified earlier, verify that

$$\begin{aligned} Q_1(x) &= 4x^3 + \frac{\sin(5x)}{5} + \sin(10x) + 10x\left(2\sin^2(5x) - 1\right), \\ Q_2(x) &= x^4 - x\sin(10x) - \frac{2\cos^2(5x)}{5} - \frac{\cos(5x)}{25} + \frac{11}{25}. \end{aligned} \tag{7}$$

With the exact formula for Q_1 and Q_2, the temperature function is simply (note this function as it will be used in all problems below!)

$$T(x) = 100x + 200xQ_2(1) - 200Q_2(x). \tag{8}$$

Remark 3. In the hints file, I show how you can verify formulas for Q_1 and Q_2 using MATLAB symbolic library. Use the codes in the hints file to complete this problem.

(ii) Plot q_{ext} and T (T is given by (8)) in the same plot. Also plot the horizontal lines $y = 80$ and $y = 40$ in the same plot. Add labels to each curves in MATLAB plot. (See hints file where all this is practically done. Try to make new changes to the codes I provide as you see fit.)

(iii) *(Optional)* Show that T in (8) satisfies ODE (1) and boundary conditions (2). That is, compute $-\kappa A\frac{\mathrm{d}^2T}{\mathrm{d}x^2}$ and show that it is equal to Aq_{ext}, and show $T(0)$ and $T(1)$ is equal to T_0 and T_1, respectively.

Problem 2 (25 marks). *Roots problem.* Find a point in the bar, i.e. x, such that the temperature $T(x)$ is 80 Celsius. I.e., solve roots problem $f(x) = 0$ with $f(x) = T(x) - 80$, where T is given by (8).

(i) Use **incremental search method** with $n = 5$ intervals and initial bracket $[0, 1]$, and find the interval $[x_l, x_u]$ that contains the root of function $f(x) = T(x) - 80$.

(ii) With interval $[x_l, x_u]$ computed in (i) above, now apply **Bisection method** to more accurately locate the root of function $f(x) = T(x) - 80$. For Bisection method, set max iteration to 100 and tolerance on relative percentage error of 0.001% .

Remark 4. Using the plot in **Problem 1**, you can verify your results. This works also for the **Problems 3 and 4**.

Problem 3 (25 marks). *Roots problem.* Find a point in the bar, i.e. x, such that the temperature $T(x)$ is 40 Celsius. I.e., solve the roots problem $f(x) = 0$ with $f(x) = T(x) - 40$.

(i) With initial guess $x_0 = 0.15$, apply the **Newton-Raphson method** to solve the roots problem $f(x) = T(x) - 40 = 0$. Use max iteration 100 and tolerance of relative percentage error 0.001%.

(ii) With initial guess $x_0 = 0.15$, apply the **Secant's method** with max iteration 100 and tolerance of relative percentage error 0.001%. For the Secant's method, use $h = 0.0001$ in approximation of a derivative.

Problem 4 (40 marks). *Optimization problem.* It is important to know what is the maximum temperature in the bar. Therefore, solve the following maximization (optimization) problem:

$$\max_{x \in [0.6, 0.9]} T(x). \tag{9}$$

In the above, I could also have $x \in [0, 1]$, but judging by the plots in **Problem 1**, I know that the maximum is somewhere in this interval $[0.6, 0.9]$ so we are good. Complete this problem in the following steps:

(i) We know that at points where the function is maximum or minimum, the derivative of a function will be zero. I.e., if at $\bar{x}$, $T(\bar{x})$ is maximum, then $\mathrm{d}T(\bar{x})/\mathrm{d}x = 0$. We use this fact to convert the maximization problem into the roots problem: Let $f(x) = \mathrm{d}T(x)/\mathrm{d}x$, then find $\bar{x}$ in the interval $[0.6, 0.9]$ such that $f(\bar{x}) = 0$. Use the **Bisection method** with the max iterations 100 and error (relative percentage error) tolerance 0.001%. Of course, for the Bisection method, use initial bracket $[0.6, 0.9]$, i.e., $x_l = 0.6$ and $x_u = 0.9$. Also, report the value of temperature T at the approximate solution $\bar{x}$ from the Bisection method.

(ii) Repeat (i) but now using the **Secant's method** with the initial guess as $x_0 = 0.7$. As before, use $h = 0.0001$ to compute the approximate derivatives in the Secant's method.