Lecture 9

Open methods

1. Fixed point iteration method
2. Newton-Raphson method
3. Secant method
4. Brent's method

Roots problem

$f(x_0) = 0$

Fixed point problem

Given a function $g: X \to (-\infty, \infty)$ find a point $x_0 \in X$ such that

$x_0 = g(x_0)$

Fixed-point iteration method

Consider following function $f: X \to Y$

$$f(x) = x - g(x)$$

where g is another function $g: X \to Y$.

Roots of function f: find $x_0 \in X$ such that

$$f(x_0) = 0 \Rightarrow x_0 - g(x_0) = 0$$

$$\Rightarrow \boxed{x_0 = g(x_0)}$$

$\Downarrow$

find x_0 such that $x_0 = g(x_0)$.

For any function f: we can always have

$$f(x) = x - g(x)$$

by defining $\boxed{g(x) := x - f(x)}$

$f(x_0) = 0$

$\Rightarrow x_0 - g(x_0) = 0$

$\Rightarrow \boxed{x_0 = g(x_0)}$

Thus for any function f: root problem can be written as "find x_0 such that $x_0 = g(x_0)$"

!!! Problem of finding x such that $x = g(x)$ is called fixed-point iteration problem

Roots problem $f(x_0) = 0$

$\Downarrow$

fixed-point problem

(i) $g(x) = x - f(x)$

(ii) find x_0 such that $x_0 = g(x_0)$

How to solve $x = g(x)$?

- Suppose x^0 is the initial guess
- then we find the next x by using $x^1 = g(x^0)$

⋮

find the x at i^{th} iteration $x^i = g(x^{i-1}) \Rightarrow$ for i=1:n x(i) =g(x(i-1)) end

we perform this iteration until error $e_a = \dfrac{|x^i - x^{i-1}|}{|x^i|} \times 100\%$

is below our tolerance.

$\Rightarrow$ Easy to implement in MATLAB

$\Rightarrow$ However, we first need to study the properties of the iterative method $x^i = g(x^{i-1})$

$x = 1$

Example 1 $f(x) = (x-1)^2$, $X = (-\infty, \infty)$, $Y = [0, \infty)$

Let $g(x) = x - f(x) = x - (x-1)^2$

Let initial guess is $x^0 = 0.5$

iteration 1 : $x^1 = g(x^0) = 0.5 - 0.25 = 0.25$ $\quad \dfrac{|0.25 - 0.5|}{|0.25|}$

iteration 2 : $x^2 = g(x^1) = 0.25 - 0.5625 = -0.3125$ $\quad$ 1.

iteration 3 : $x^3 = g(x^2) = -0.3125 - (1.3125)^2 = -2.035$

iteration 4 : $x^4 = g(x^3) = -2.035 - (-2.035-1)^2 = -11.25$

⋮

diverging

Let initial guess $x^0 = 1.1$ $\qquad \dfrac{|1.09 - 1.1|}{|1.09|}$

Then <u>iteration 1</u> : $x^1 = g(x^0) = 1.1 - 0.01 = 1.09$

<u>iteration 2</u> : $x^2 = g(x^1) = 1.09 - (0.09)^2 = 1.0819$

<u>iteration 3</u> : $x^3 = g(x^2) = 1.0752$

$\vdots$

Converging to $x_0 = 1$

<u>Example 2 :</u> $f(x) = x - \cos(x)$, $X = (-\infty, \infty)$, $Y = (-\infty, \infty)$

Then $g(x) = x - f(x) = \cos(x)$

<u>Initial guess</u> : $x^0 = 0.5$

<u>iter. 1</u> : $x^1 = g(x^0) = \cos(0.5) = 0.8776$

<u>iter. 2</u> : $x^2 = g(x^1) = \cos(0.8776) = 0.639$

<u>iter. 3</u> : $x^3 = \cos(0.639) = 0.803$

<u>iter. 4</u> : $x^4 = 0.695$

<u>iter. 5</u> : $x^6 = 0.768$

<u>iter. 6</u> : $x^7 = 0.7193$

<u>iter. 7</u> : $x^8 = 0.752$

$\vdots$

$x^{i} = 0.7388$

Converging

To understand how fixed-point iteration works

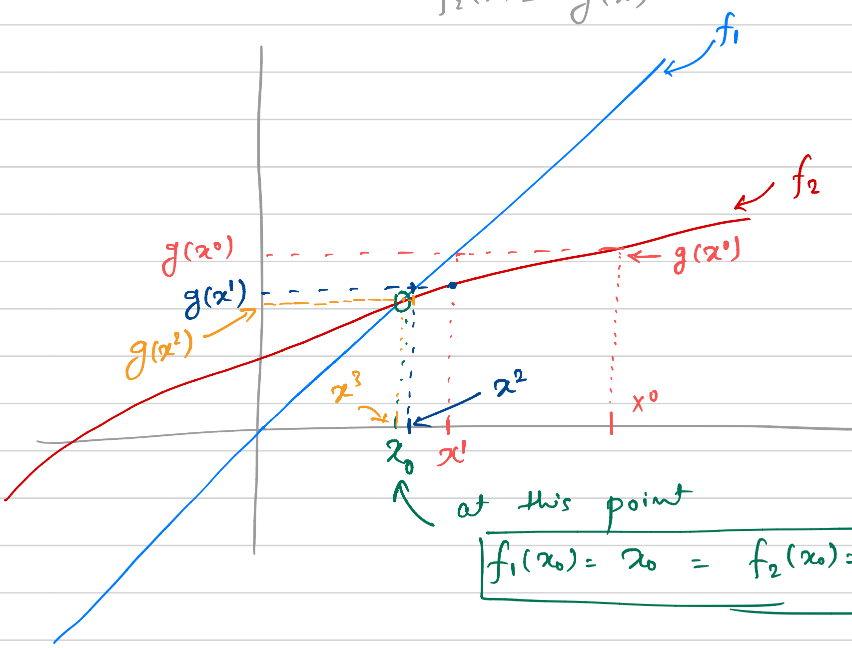
let $f_1(x) = x$

$f_2(x) = g(x)$

The solution of $x = g(x)$ problem is a point x_0 such that

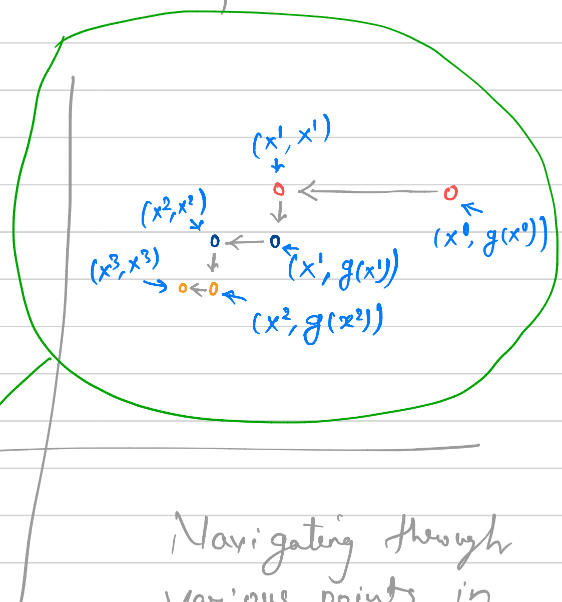
$$f_1(x_0) = f_2(x_0)$$

I.e. point at which two functions intersect



at this point

$$\boxed{f_1(x_0) = x_0 = f_2(x_0) = g(x_0)}$$



Navigating through various points in fixed-point iteration method

Plot our iteration steps:

iter. 1 : $x^1 = g(x^0)$

iter 2 : $x^2 = g(x^1)$

iter 3 : $x^3 = g(x^2)$

⋮

Can we say more about this particular example ?

$\Downarrow$

Given a point x^i, we compute $x^{i+1} = g(x^i)$

we see that

$$\left\{ \begin{array}{l} x^1 = g(x^0) < x^0 \\ x^2 = g(x^1) < x^1 \\ x^3 = g(x^2) < x^2 \\ \vdots \end{array} \right.$$

Generally, for any $x > x_0$

$g(x) < x$

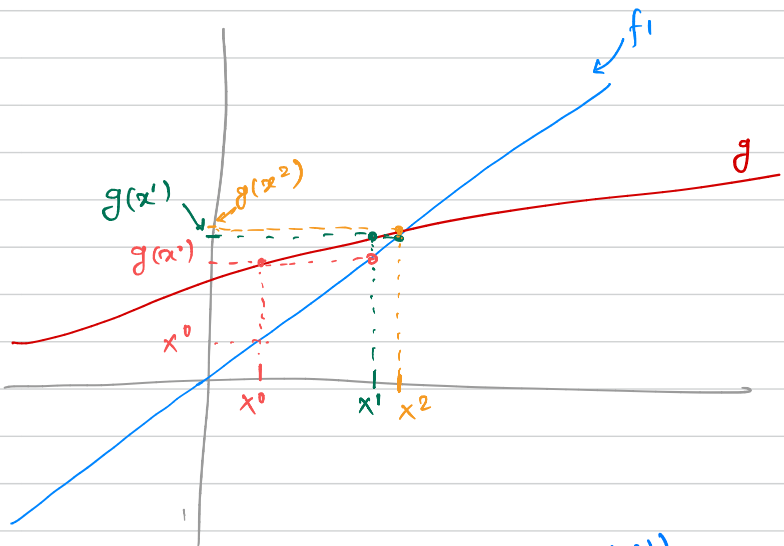
where x_0 is the true solution of $x = g(x)$

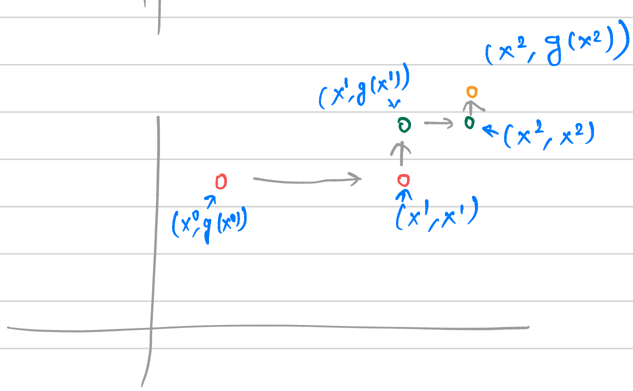
Previously, we considered a function g such that

$g(x) < x$ for any $x > x_0$ ← true solution of $x = g(x)$

→ for such a function, we see that in each iteration we got closer to true solution x_0.

Let us see now the case when x^0 (initial guess) is on the left side of true solution:





⇒ In this case also we see that in each iteration we are getting closer to true solution x_0

⇓

"we observe"

$x^1 = g(x^0) > x^0$

$x^2 = g(x^1) > x^1$

$x^3 = g(x^2) > x^2$

↓

for $x < x_0$ (where x_0 is the true solution), we have

$g(x) > x$

initial guess

true solution

Thus if

(i) we start from right side of x_0, i.e. $x^0 > x_0$

we want $g(x) < x$ for any $x > x_0$,

So that successive iterations will reduce

x^i trying to get closer to x_0

I.e. need $g(x) < x$ for $x > x_0$ so

that we get

$$x^0 > x^1 > x^2 > x^3 \cdots > x_0$$

(ii) We start from left side of x_0, i.e. $x^0 < x_0$,

then we want $g(x) > x$ for any $x < x_0$,

So that successive iterations will increase x^i

taking it closer to x_0

I.e. need $g(x) > x$ for $x < x_0$ so

$$x^0 < x^1 < x^2 < x^3 < \cdot \cdot < x_0$$

$\Downarrow$

What happens when g does not have this property?

Is it still possible to converge to x_0?

Error in fixed point iteration method

let x_0 is such that $\boxed{x_0 = g(x_0)}$ (So x_0 is the true solution)

Let E_t^i := true error at iteration i

$= x^i - x_0$

Since $x^i = g(x^{i-1})$ ← our iteration method!

$\Rightarrow E_t^i = x^i - x_0$

(1) $= g(x^{i-1}) - x_0$

$$\Rightarrow \boxed{E_t^i = g(x^{i-1}) - g(x_0)}$$

(∵ x_0 is true solution so $x_0 = g(x_0)$)

We know from Taylor's series expansion

$$\boxed{f(x) = f(y) + \frac{df}{dy}(y)(x-y) + \frac{1}{2!}\frac{d^2f}{dy^2}(y)(x-y)^2 + \dots + \frac{1}{n!}\frac{d^nf}{dy^n}(y)(x-y)^n + \dots}$$

$2! = 2$

$$\Rightarrow f(x) = f(y) + \frac{df}{dy}(y)(x-y) + \frac{1}{2!}\frac{d^2f}{dy^2}(y)(x-y)^2 + \dots + \frac{1}{n!}\frac{d^nf}{dy^n}(z)(x-y)^n$$

↑ there exists $z \in X$ and z depends on choice of x, y, n

$$\Rightarrow f(x) = f(y) + \frac{df(z)}{dy}(x-y)$$

↑ there exist $z \in X$

z depends on x and y

We can write

$$\boxed{g(x^{i-1}) = g(x_0) + \frac{dg(z)}{dy}(x^{i-1} - x_0)} \quad (2)$$

z is not known and generally

z will depend on x^{i-1} and x_0

$$\boxed{\frac{|E_t^i|}{|E_t^{i-1}|} < 1 \Rightarrow |E_t^i| < |E_t^{i-1}| <} |E_t^{i-2}| \dots < |E_t^1|$$

Thus combining (1) and (2)

$$E_t^i = \frac{dg(z)}{dy}\underbrace{(x^{i-1} - x_0)}_{E_t^{i-1}}$$

$$(3) \Rightarrow \boxed{E_t^i = \frac{dg(z)}{dy} E_t^{i-1}} \Longrightarrow \boxed{\frac{|E_t^i|}{|E_t^{i-1}|} \le \left|\frac{dg(z)}{dy}\right| \le M}$$

Suppose such a number M exist

$$\Updownarrow$$

$\boxed{g : X \to (-\infty, \infty) \text{ is a function such that at any point } x \in X, \left|\frac{dg}{dy}(x)\right| < 1}$

Equation (3) is very important result and provides mathematical reasoning as to when error will decrease in successive iterations and when it will increase

$$\Downarrow$$

when method will converge and when it will diverge

We want x^i to get closer and closer to x_0 with increasing i

I.e. $|E_t^1| > |E_t^2| > |E_t^3| > \dots > |E_t^i| > \dots$

OR $1 > \frac{|E_t^2|}{|E_t^1|}, \quad 1 > \frac{|E_t^3|}{|E_t^2|}, \quad \dots \quad 1 > \frac{|E_t^i|}{|E_t^{i-1}|}$

Since $\dfrac{|\epsilon_t^{i}|}{|\epsilon_t^{i-1}|} \leq \left|\dfrac{dg(z)}{dy}\right|$ for any $z \in X$

Convergence is guaranteed if $\left|\dfrac{dg(x)}{dy}\right| < 1$ for all points $x \in X$

$$\Downarrow$$

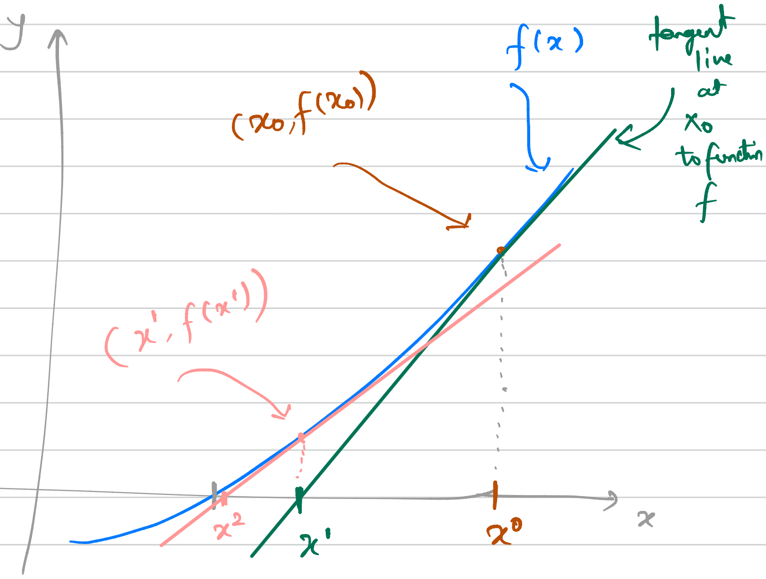
$$\frac{|\epsilon_t^{i}|}{|\epsilon_t^{i-1}|} < 1$$

$$\Downarrow$$

Thus fixed-point iteration converges surely if slope of function g at any point $x \in X$ is below 1

Newton - Raphson method

Given a function f



— initial guess x^0

— find a tangent line at x_0 to function f

$y = y(x) = m x + c$

(i) it is a tangent to function f at $(x_0, f(x_0))$

$$\boxed{m = f'(x_0)}$$

x-coordinate

y-coordinate

(ii) passes through $(x_0, f(x_0))$

$\Rightarrow f(x_0) = m x_0 + c = f'(x_0) x_0 + c$

$$\Rightarrow \boxed{c = f(x_0) - x_0 f'(x_0)}$$

$$y(x) = f'(x_0) x + f(x_0) - x_0 f'(x_0)$$

$$\boxed{y(x) = f'(x_0)(x - x_0) + f(x_0)}$$

— find a point x' s.t $y(x') = 0$ (line y intersects x-axis at point x')

$\Rightarrow f'(x^0)(x' - x^0) + f(x^0) = 0$

$$\Rightarrow \boxed{x' = x^0 - \frac{f(x^0)}{f'(x^0)}}$$

— find x^2

(i) find a new line that is tangent to f at x^1

and that passes through point $(x^1, f(x^1))$

$$y(x) = f'(x^1)(x - x^1) + f(x^1)$$

(ii) find x^2 s.t $y(x^2) = 0$

$$\Rightarrow \boxed{x^2 = x^1 - \frac{f(x^1)}{f'(x^1)}}$$

— f at iteration i

$$\boxed{x^i = x^{i-1} - \frac{f(x^{i-1})}{f'(x^{i-1})}}$$