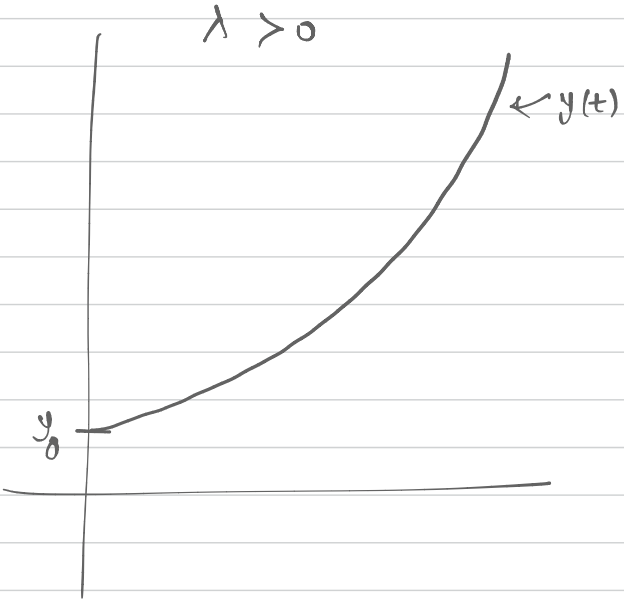
Lecture 39

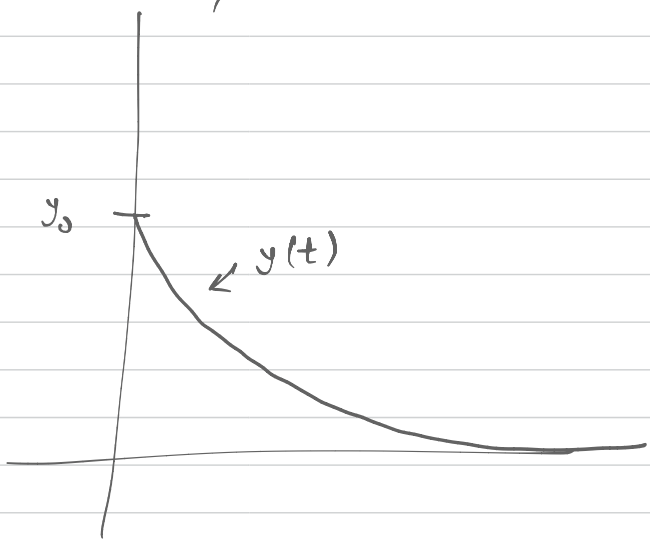
- Errors due to approximation (consistency error)
- Stability : whether solution is diverging or not

$$\frac{dy}{dt} = \lambda y \ , \quad y(0) = y_0 \Rightarrow y(t) = y_0 e^{\lambda t}$$

$\lambda > 0$



$\lambda < 0$



Stability of numerical method makes sense when the problem is stable.

Forward Euler	Backward Euler
$t_1, t_2, \dots, t_N$	
$\frac{dy}{dt}(t_i) = \lambda y(t_i)$	$\frac{dy}{dt}(t_i) = \lambda y(t_i)$
$\Rightarrow \frac{y(t_{i+1}) - y(t_i)}{\Delta t} = \lambda y(t_i)$	$\Rightarrow \frac{y(t_i) - y(t_{i-1})}{\Delta t} = \lambda y(t_i)$
$\Rightarrow y(t_{i+1}) = y(t_i) + \Delta t \lambda y(t_i)$	$\Rightarrow y(t_i) = y(t_{i-1}) + \Delta t \lambda y(t_i)$

$$\Rightarrow \boxed{y(t_{i+1}) = (1+\Delta t\lambda)\, y(t_i)}$$

$a = 1 + \Delta t \lambda$

$\Rightarrow y(t_{i+1}) = a\, y(t_i)$

$y(t_i) = a\, y(t_{i-1})$

$y(t_{i-1}) = a\, y(t_{i-2})$

$$\begin{aligned} y(t_{i+1}) &= a\, y(t_i) \\ &= a^2 y(t_{i-1}) \\ &= a^3 y(t_{i-2}) \\ &= \vdots \end{aligned}$$

$$\boxed{y(t_{i+1}) = a^{i+1} y_0}$$

$T = 1$

$\Delta t = 10^{-6}$

$i = 1, 2, \dots, 10^6$

$i \longrightarrow \infty$

what happens to a^i

Q. does $a^i \longrightarrow \infty$

or $a^i \longrightarrow -\infty$

or $a^i \longrightarrow 0$

or $a^i \longrightarrow M$, $|M| < \infty$

$\Rightarrow (1 - \Delta t \lambda)\, y(t_i) = y(t_{i-1})$

$$\Rightarrow \boxed{y(t_i) = \frac{1}{1-\Delta t \lambda}\, y(t_{i-1})}$$

$b = \dfrac{1}{1 - \Delta t \lambda}$

then $y(t_i) = b\, y(t_{i-1})$

$= b^2 y(t_{i-2})$

$\vdots$

$$\boxed{y(t_i) = b^i y_0}$$

when $i \longrightarrow \infty$

$b^i \longrightarrow ?$

$b = \dfrac{1}{1 - \Delta t \lambda}$

assume we have stable ODE

↳ means that $\lambda < 0$

but since $\Delta t > 0$

$0 < b < 1$

↳ $b^i \longrightarrow 0$

$a = 1 + \Delta t \lambda$

- $\Delta t > 0$

- if $\lambda > 0$

  then $a = 1 + \Delta t \lambda > 1$

  $\downarrow$

  therefore $a^i \longrightarrow \infty$

- if $\lambda < 0$

  then $a = 1 + \Delta t \lambda$

  (i) $a < 1$

  but (ii) $a < -1$ → diverging

  $\downarrow$

  $a^i \longrightarrow -\infty$ when i is odd

  $a^i \longrightarrow \infty$ when i is even

  → converging

  (iii) $-1 < a < 1$

  $\downarrow$

  $a^i \longrightarrow 0$

  (iv) if $a = 1$, $a^i \longrightarrow 1$

  (v) if $a = -1$, $a^i \longrightarrow -1$ if i is odd

  $\longrightarrow 1$ if i is even

$a = 1 + \Delta t \lambda \qquad (\lambda < 0)$

want Δt s.t.

$$\boxed{-1 \leq a \leq 1}$$

$$\Downarrow$$

$$-1 \leq a = 1 + \Delta t \lambda$$

$$\Rightarrow -\Delta t \lambda \leq 2$$

$$\Rightarrow \boxed{\Delta t \leq \frac{-2}{\lambda}}$$

condition for stability of forward Euler

- Approximating system of ODEs (1st order ODEs)

- $\dfrac{dy}{dt} = Ay, \qquad y = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix},$ A is $n \times n$ matrix

- $y(0) = y_0$

- forward Euler discretization

pick i^{th} equation in $\dfrac{dy}{dt} = Ay$, $A = [a_{ij}]$

$$\frac{d\, y_i(t)}{dt} = \sum_{j=1}^{n} a_{ij}\, y_j(t)$$

$$\downarrow$$

$$\frac{y_i(t_{k+1}) - y_i(t_k)}{\Delta t} = \sum_{j=1}^{n} a_{ij}\, y_j(t_k)$$

$$\Rightarrow \quad \boxed{y_i(t_{k+1}) = y_i(t_k) + \Delta t \sum_{j=1}^{n} a_{ij}\, y_j(t_k)}$$

$$\Big\Downarrow \text{ matrix notation}$$

$$\boxed{y(t_{k+1}) = y(t_k) + \Delta t\, A\, y(t_k)}$$

$$\Big\Downarrow$$

where I is identity matrix

$$\boxed{y(t_{k+1}) = (I + \Delta t A)\, y(t_k)}$$

- Backward Euler discretization

pick i^{th} equation

$$\frac{dy_i}{dt}(t) = \sum_{j=1}^{n} a_{ij}\, y_j(t)$$

$$\downarrow$$

$$\frac{dy_i}{dt}(t_{k+1}) = \sum_{j=1}^{n} a_{ij}\, y_j(t_{k+1})$$

$$\Rightarrow \quad \frac{y_i(t_{k+1}) - y_i(t_k)}{\Delta t} = \sum_{j=1}^{n} a_{ij}\, y_j(t_{k+1})$$

$$\Rightarrow \quad \boxed{y_i(t_{k+1}) = y_i(t_k) + \Delta t \sum_{j=1}^{n} a_{ij}\, y_j(t_{k+1})}$$

$$\Big\Downarrow \text{ matrix representation}$$

$$\boxed{y(t_{k+1}) = y(t_k) + \Delta t\, A\, y(t_{k+1})}$$

$\Rightarrow \quad y(t_{k+1}) - \Delta t\, A\, y(t_{k+1}) = y(t_k)$

$\Rightarrow \quad \boxed{(I - \Delta t A)\, y(t_{k+1}) = y(t_k)}$

define $\quad J = I - \Delta t A, \quad b = y(t_k), \quad x = y(t_{k+1})$

solve $\Rightarrow \quad \boxed{Jx = b}$

$[L_J, U_J] = lu(J)$