Lecture 38

ODE: $\frac{dy}{dt} = f(t, y)$, $y(0) = y_0$

notation: $y_i = y(t_i)$, $f_i = f(t_i, y_i)$

- Single step methods

(1.) Forward Euler Method

$$y_{i+1} = y_i + \Delta t\, f_i$$

(2.) Backward Euler Method

$$y_{i+1} = y_i + \Delta t\, f_{i+1}$$

(3.) Heun's Method (Trapezoidal Method)

$$y_{i+1}^0 = y_i + \Delta t\, f_i \qquad \text{(predictor)}$$

$$y_{i+1} = y_i + \frac{f_i + f(t_{i+1}, y_{i+1}^0)}{2}\Delta t \qquad \text{(corrector)}$$

(4.) Midpoint Method

$$y_{i+1/2} = y_i + f_i \frac{\Delta t}{2}$$

$$y_{i+1} = y_i + f(t_{i+1/2}, y_{i+1/2})\, \Delta t$$

Notation

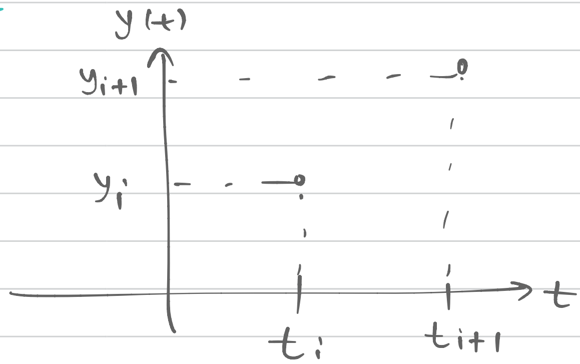
$t_{i+1/2} = t_i + \Delta t/2$

$y_{i+1/2} = y(t_{i+1/2})$

- Errors in forward Euler method

(t_i, y_i) ✓ , $f(t_i, y_i)$ ✓

(t_{i+1}, y_{i+1}) ?



Assume: f is just a function of t.

$f(t_i, y_i)$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\frac{df}{dt}(t_i)$

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)\frac{dy}{dt}(t_i) + \frac{(t_{i+1}-t_i)^2}{2!}\frac{d^2y}{dt^2}(t_i)$$

$$+ \cdots + \frac{(t_{i+1}-t_i)^{n-1}}{(n-1)!}\frac{d^{n-1}y}{dt^{n-1}}(t_i) \qquad \frac{d^{n-2}f}{dt^{n-2}}(t_i)$$

$$+ \frac{(t_{i+1}-t_i)^n}{n!}\frac{d^n y}{dt^n}(t_i)$$

$$+ \cdots$$

note $\quad t_{i+1} - t_i = \Delta t \qquad (\Delta t_i)$

$$y^{(k)}(t) = \frac{d^k y}{dt^k}(t)$$

$$\frac{dy}{dt}(t_i) = f(t_i)$$

$$\frac{d^k y}{dt^k}(t_i) = f^{(k-1)}(t_i), \qquad k \geq 1$$

$$\Rightarrow \quad y_{i+1} = y_i + \Delta t\, f_i + \frac{\Delta t^2}{2!} f_i' + \cdots + \frac{\Delta t^{n-1}}{(n-1)!} f_i^{(n-1)}$$

$$+ \frac{\Delta t^n}{n!} f_i^{(n)} + \cdots$$

another version (truncated)

$$y_{i+1} = y_i + \Delta t \frac{dy}{dt}(t_i) + \cdots + \frac{\Delta t^{n-1}}{(n-1)!}\frac{d^{n-1}y}{dt^{n-1}}(t_i)$$

$$+ \frac{\Delta t^n}{n!}\frac{d^n y}{dt^n}(z_n)$$

for some $z_n \in [t_i, t_{i+1}]$

another version with O notation

$$y_{i+1} = y_i + \Delta t \frac{dy}{dt}(t_i) + \cdots + \frac{\Delta t^{n-1}}{(n-1)!} \frac{d^{n-1}y}{dt^{n-1}}(t_i)$$

$$+ O(\Delta t^n)$$

$$\boxed{y_{i+1} = y_i + \Delta t f_i} + \boxed{\cdots + \frac{\Delta t^{n-1}}{(n-1)!} f_i^{(n-2)} + O(\Delta t^n)}$$

forward Euler

source of errors

$$O(\Delta t^2)$$

$$= \frac{\Delta t^2}{2!} f_i^{(1)} + \frac{\Delta t^3}{3!} f_i^{(2)} + \cdots + \frac{\Delta t^{n-1}}{(n-1)!} f_i^{(n-2)}$$

$$+ \cdots$$

$$y_{i+1} = y_i + \Delta t \frac{dy}{dt}(t_i) + \frac{\Delta t^2}{2!} \frac{d^2y}{dt^2}(\eta), \quad \eta \in [t_i, t_{i+1}]$$

$$O(\Delta t^2) = \frac{\Delta t^2}{2!} \frac{d^2y}{dt^2}(\eta)$$

$$= \frac{\Delta t^2}{2!} f^{(1)}(\eta)$$

$$\Rightarrow \boxed{y_{i+1} = y_i + \Delta t f_i} + \boxed{O(\Delta t^2)}$$

Local errors : error in single step

$$\underbrace{(t_i, y_i) \longrightarrow (t_{i+1}, y_{i+1})}_{\text{one step}}$$

$$\boxed{\text{the error in this step is } O(\Delta t^2)}$$

Global error: error over all steps

$(t_1, y_1) \longrightarrow (t_2, y_2) \longrightarrow (t_3, y_3) \longrightarrow \cdots$

Δt is time step, T is final time,

then $\frac{T}{\Delta t}$ is the number of time steps.

$$\boxed{\begin{aligned} \text{global error} &\approx \frac{T}{\Delta t} \, O(\Delta t^2) \\ &= O(\Delta t) \end{aligned}}$$

- Error in backward Euler

$$y(t_i) = y(t_{i+1}) + \frac{(t_i - t_{i+1})}{1} \frac{dy}{dt}(t_{i+1})$$

$$+ \frac{(t_i - t_{i+1})^2}{2!} \frac{d^2y}{dt^2}(\eta), \qquad \eta \in [t_i, t_{i+1}]$$

$$\Rightarrow \quad y_i = y_{i+1} - \Delta t \, f_{i+1} + \underbrace{\frac{\Delta t^2}{2!} f^{(1)}(\eta)}_{O(\Delta t^2)}$$

$$\Rightarrow \quad \boxed{y_{i+1} = y_i + \Delta t \, f_{i+1}} + \boxed{O(\Delta t^2)}$$

backward Euler $\qquad$ source of error

$\therefore$ local error is $O(\Delta t^2)$

$\therefore$ global error is $O(\Delta t)$

· Error in Heun's method

(t_i, y_i) ✓

(t_{i+1}, y_{i+1}) ?

$$\underbrace{\frac{dy}{dt} = f(t)}_{\text{integrate over } [t_i, t_{i+1}]}$$

$$\int_{t_i}^{t_{i+1}} \frac{dy}{dt}\, dt = \int_{t_i}^{t_{i+1}} f(t)\, dt$$

$$\Rightarrow \quad y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} f(t)\, dt$$

$$\Rightarrow \quad y_{i+1} = y_i + \underbrace{\int_{t_i}^{t_{i+1}} f(t)\, dt}_{\text{Trapezoidal rule}}$$

$$\approx \frac{f(t_i) + f(t_{i+1})}{2} (t_{i+1} - t_i)$$

$$= \frac{f_i + f_{i+1}}{2} \Delta t$$

$$\Rightarrow \quad y_{i+1} \approx y_i + \frac{f_i + f_{i+1}}{2} \Delta t$$

$$\int_a^b f(x)\,dx = \frac{f(a) + f(b)}{2}(b-a) + \underbrace{\frac{1}{12} f^{(2)}(m)\,(b-a)^3}_{\text{error}}$$

$$\boxed{y_{i+1} = y_i + \frac{f_i + f_{i+1}}{2}\,\Delta t} + \boxed{\underbrace{\frac{1}{12} f^{(2)}(x_m)\,(\Delta t)^3}_{O(\Delta t^3)}}$$

So local error is $O(\Delta t^3)$

global error is $O(\Delta t^2)$