Lecture 36

- **Numerical methods for ODE**

- $\dfrac{dy}{dt}(t) = f(t, y(t))$, $0 \le t \le T$

- $y(0) = y_0$

> - $f(t, y(t)) = g(t)$
> - $f(t, y(t)) = a\, y(t)$
> - $f(t, y(t)) = g(t) + a\, y(t)^2$

Two classes of methods

> notation
>
> $y_i = y(t_i)$

(i) single-step method

if I know (t_i, y_i)

then

$$y_{i+1} = y(t_{i+1}) = g(t_i, y_i, t_{i+1} - t_i)$$

(ii) multi-step method

if I have $(t_{i-2}, y_{i-2}), (t_{i-1}, y_{i-1}), (t_i, y_i)$

then

$$y_{i+1} = g(t_i, t_{i-1}, t_{i-2}, y_i, y_{i-1}, y_{i-2})$$

- **Single step method**

(i) Euler's method

- forward Euler / explicit Euler method
- backward Euler / implicit Euler method

(ii) Heun's method

(iii) Midpoint method

(iv) Runge - Kutta method

Euler method

- $\frac{dy}{dt}(t_2) = f(t_2, y_2)$

- $\frac{dy}{dt}(t_3) = f(t_3, y_3)$

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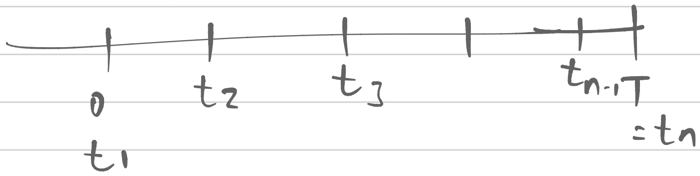
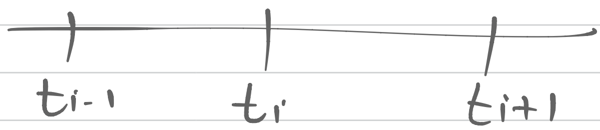
- $\frac{dy}{dt}(t_i) = f(t_i, y_i)$



- $\Delta t_i = t_{i+1} - t_i$
- $y_i = y(t_i)$



- $\frac{dy}{dt}(t_i) \approx \frac{y_{i+1} - y_i}{\Delta t_i}$ → forward difference

- $\frac{dy}{dt}(t_i) \approx \frac{y_i - y_{i-1}}{\Delta t_{i-1}}$ → backward difference

- Using forward difference approximation

$$\frac{y_{i+1} - y_i}{\Delta t_i} = f(t_i, y_i)$$

$$\Rightarrow \boxed{y_{i+1} = y_i + \Delta t_i \, f(t_i, y_i)}$$

- Using backward difference

$$\frac{y_i - y_{i-1}}{\Delta t_{i-1}} = f(t_i, y_i)$$

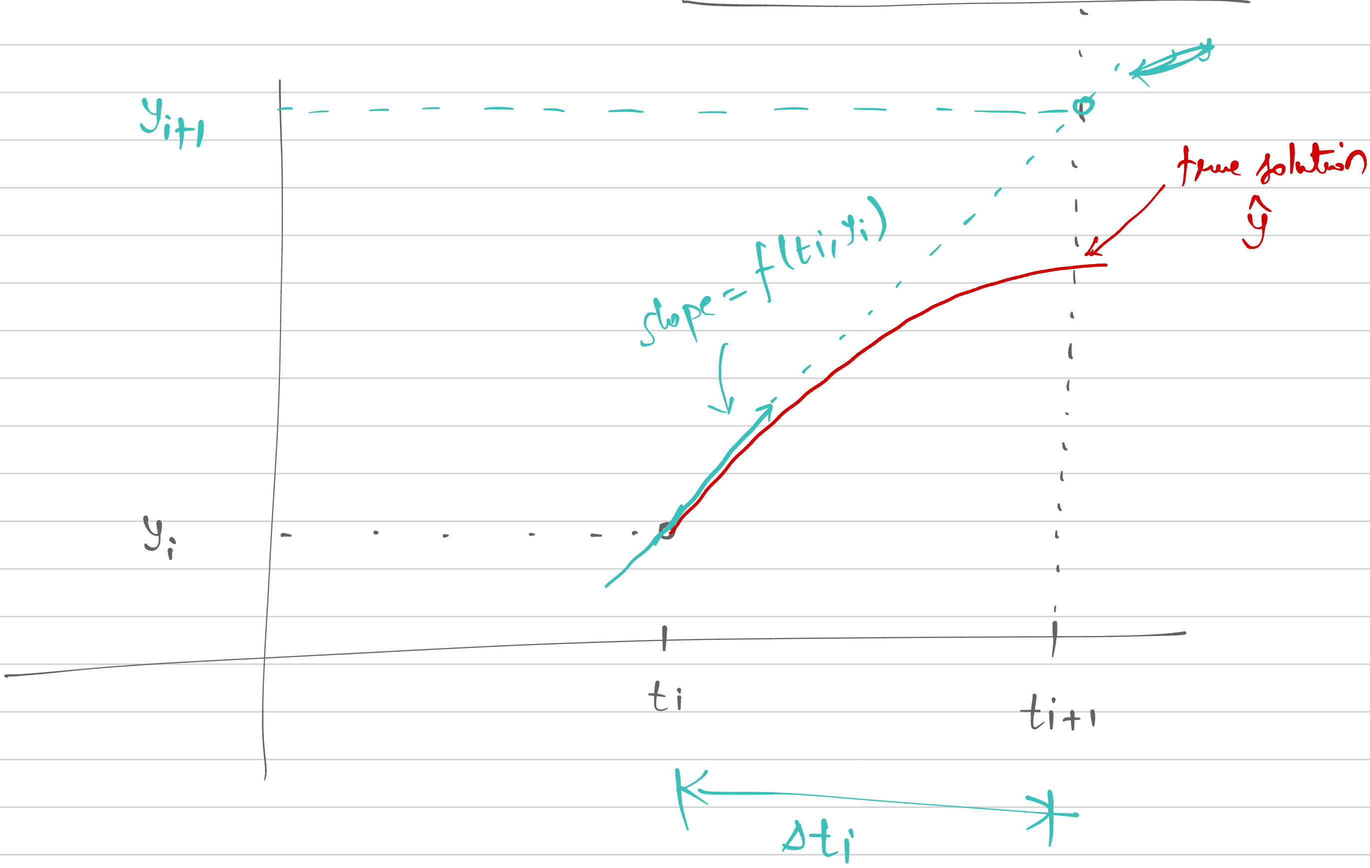
$$\Rightarrow \boxed{y_i = \underbrace{y_{i-1} + \Delta t_{i-1} f(t_i, y_i)}}$$

$$g(y_i) = y_{i-1} + \Delta t_{i-1} f(t_i, y_i)$$

$$\Rightarrow \boxed{y_i = g(y_i)}$$ fixed point iteration

- Heun's method

$$\boxed{y_{i+1} = y_i + \Delta t_i f(t_i, y_i)}$$



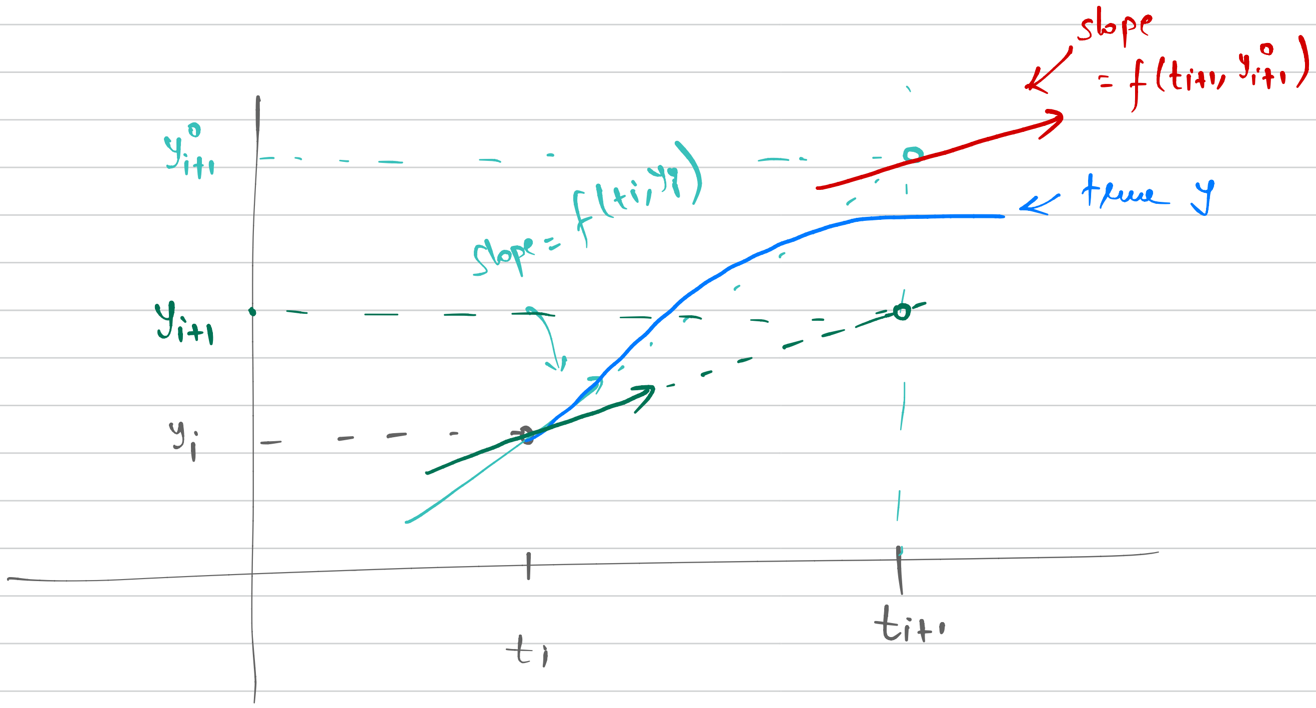
(t_i, y_i) & (t_{i+1}, y_{i+1}) are on straight line

- predicator / estimator

$$\boxed{y^0_{i+1} = y_i + \Delta t_i \, f(t_i, y_i)}$$

- Corrector

$$y_{i+1} = y_i + \Delta t_i \left(\frac{f(t_i, y_i) + f(t_{i+1}, y^0_{i+1})}{2} \right)$$



- Midpoint method

Given (t_i, y_i)

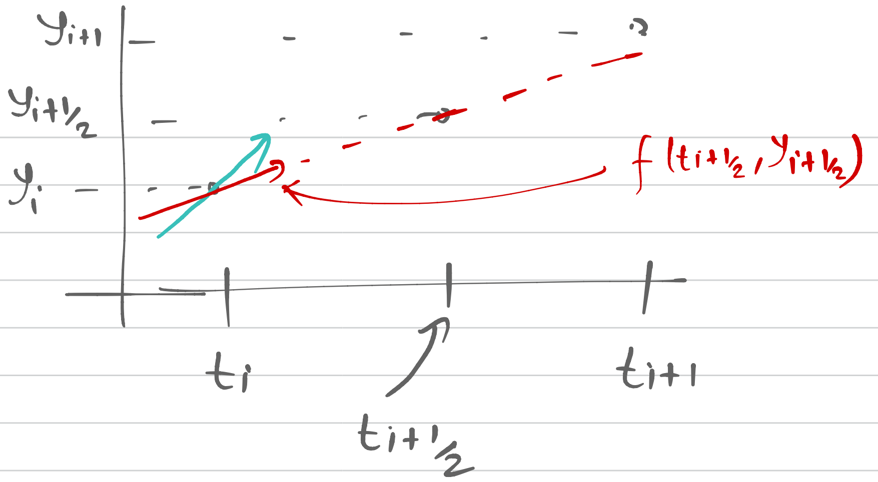
- $t_{i+1/2} = \frac{t_i + t_{i+1}}{2}$
  $= t_i + \frac{1}{2} \Delta t_i$

  $\Delta t_i = t_{i+1} - t_i$
- $y_{i+1/2} = y(t_{i+1/2})$

(i) $$\underbrace{y_{i+1/2} = y_i + \frac{\Delta t_i}{2} f(t_i, y_i)}$$

$$\frac{dy}{dt}(t_i) = f(t_i, y_i)$$

$$\approx \frac{y(t_{i+1/2}) - y(t_i)}{\Delta t_i / 2} = f(t_i, y_i)$$



(ii)

$$y_{i+1} = y_i + \Delta t_i \, f(t_{i+1/2}, y_{i+1/2})$$

Compare with $y_{i+1} = y_i + \Delta t_i \, f(t_i, y_i)$

$$\frac{dy}{dt} = f(t, y(t)), \quad t_i \le t \le t_{i+1}$$

$$\int_{t_i}^{t_{i+1}} \frac{dy}{dt}\, dt = \int_{t_i}^{t_{i+1}} f(t, y(t))\, dt$$

$$\Rightarrow y_{i+1} - y_i \approx f(t_{i+1/2}, y_{i+1/2})\, \Delta t_i$$