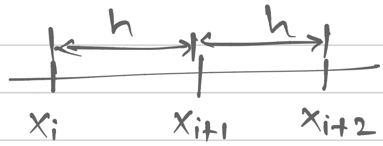
Lecture 35

• Approximation of derivatives

① Use Taylor's series to build higher order approximations (uniform discretization)

Table 21.3

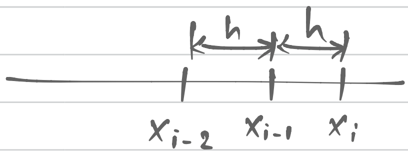


$$\boxed{f_i = f(x_i)}$$

- $f'(x_i) = \dfrac{f_{i+1} - f_i}{h} + O(h)$
- $f'(x_i) = \dfrac{-f_{i+2} + 4f_{i+1} - 3f_i}{2h} + O(h^2)$
- $f''(x_i) = \dfrac{f_{i+2} - 2f_{i+1} + f_i}{h^2} + O(h)$
- $f''(x_i) = \dfrac{-f_{i+3} + 4f_{i+2} - 5f_{i+1} - f_i}{h^2} + O(h^2)$

→ **forward difference**

Table 21.4



backward difference

- $f'(x_i) = \dfrac{f_i - f_{i-1}}{h} + O(h)$
- $f'(x_i) = \dfrac{3f_i - 4f_{i-1} + f_{i-2}}{2h} + O(h^2)$
- $f''(x_i) = \dfrac{f_i - 2f_{i-1} + f_{i-2}}{h^2} + O(h)$
- $f''(x_i) = (2f_i - 5f_{i-1} + 4f_{i-2} - f_{i-3})/h^2 + O(h^2)$

(2) Richardson's extrapolation

$$D[h_1] = \frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2h_1}$$

$$x_{i+2} - x_{i+1} = x_{i+1} - x_i = h_1$$

$$D[h_2] = \frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2h_2}$$

$$x_{i+2} - x_{i+1} = x_{i+1} - x_i = h_2$$

$$D = \frac{df}{dx}(x_i) = D[h_1] + E[h_1]$$

$$= D[h_2] + E[h_2]$$

Assume

$$E[h_1] = C\, h_1^2$$

$$E[h_2] = C\, h_2^2$$

$$D \approx D[h_2] + \frac{D[h_2] - D[h_1]}{\frac{h_1^2}{h_2^2} - 1}$$

$\downarrow$

$O(h^4)$

$I[h_1]$, $I[h_2]$

$\downarrow$ $\downarrow$

$O(h_1^2)$ $O(h_2^2)$

$$I \approx I[h_2] + \frac{I[h_2] - I[h_1]}{\frac{h_1^2}{h_2^2} - 1}$$

$\downarrow$

$O(h^4)$

- $I = I[h_1] + E[h_1]$
- $I = I[h_2] + E[h_2]$

$$E[h_1] = C\, h_1^2$$

$$E[h_2] = C\, h_2^2$$

(3) Using interpolation to get approximation

(i) not restricted to uniform discretization

(ii) compute approximate derivatives at any point in the interval

Example: (A) Linear interpolation

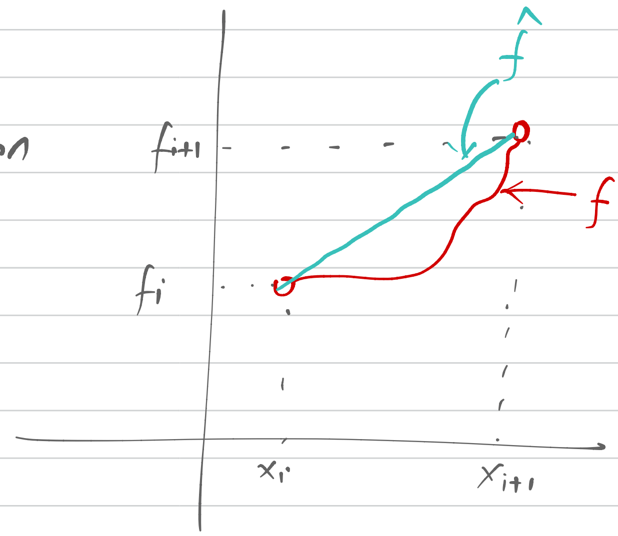
$$(x_i, f_i), \quad (x_{i+1}, f_{i+1}), \qquad f_i = f(x_i)$$

model

$$\hat{f}(x) = z(x)\, a$$

for Lagrange polynomial interpolation

$$a = \begin{bmatrix} f_i \\ f_{i+1} \end{bmatrix}$$



$$z(x) = [L_1(x), \quad L_2(x)]$$

$$L_1(x) = \frac{(x - x_{i+1})}{(x_i - x_{i+1})}$$

$$L_2(x) = \frac{(x - x_i)}{(x_{i+1} - x_i)}$$

- $$\frac{df}{dx}(x) \approx \frac{d\hat{f}}{dx}(x) = \frac{d}{dx}\left(a_1 L_1(x) + a_2 L_2(x)\right)$$

$$= a_1 \frac{dL_1}{dx}(x) + a_2 \frac{dL_2}{dx}(x)$$

$$= \left[\frac{dL_1}{dx}(x), \quad \frac{dL_2}{dx}(x)\right] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= \frac{dz}{dx}(x)\; a$$

$$\frac{d}{dx} z(x) = \left[\frac{dL_1}{dx}(x), \quad \frac{dL_1}{dx}(x)\right]$$

$$= \left[\frac{-1}{x_{i+1} - x_i}, \quad \frac{1}{x_{i+1} - x_i}\right]$$

$$\Rightarrow \frac{df}{dx} \approx \frac{d\hat{f}}{dx} = \frac{1}{\underbrace{(x_{i+1}-x_i)}_{h}} [-1, \; 1] \begin{bmatrix} f_i \\ f_{i+1} \end{bmatrix}$$

$$\boxed{\frac{df}{dx}(x) \approx \frac{d\hat{f}}{dx}(x) = \frac{f_{i+1} - f_i}{h}}$$

(B) Quadratic interpolation

(x_i, f_i), (x_{i+1}, f_{i+1}), (x_{i+2}, f_{i+2})

<u>model</u> $\hat{f}(x) = z(x)\, a$, $\quad a = \begin{bmatrix} f_i \\ f_{i+1} \\ f_{i+2} \end{bmatrix}$, $\quad z = [L_1, L_2, L_3]$

$$L_1 = \frac{(x - x_{i+1})(x - x_{i+2})}{(x_i - x_{i+1})(x_i - x_{i+2})}$$

$L_2 =$ ——

$L_3 =$ ——

- $$\frac{df}{dx}(x) \approx \frac{d\hat{f}}{dx}(x) = \left[\frac{dL_1}{dx}, \frac{dL_2}{dx}, \frac{dL_3}{dx}\right] \begin{bmatrix} f_i \\ f_{i+1} \\ f_{i+2} \end{bmatrix}$$

- $$\frac{d^2f}{dx^2}(x) \approx \frac{d^2\hat{f}}{dx^2}(x) = \left[\frac{d^2L_1}{dx^2}, \frac{d^2L_2}{dx^2}, \frac{d^2L_3}{dx^2}\right] \begin{bmatrix} f_i \\ f_{i+1} \\ f_{i+2} \end{bmatrix}$$

$x = x_i$

• Numerically Solving Ordinary Differential Equation

• Order

• $\frac{dy}{dt} = f(t, y(t))$ → 1st order ODE

• $\frac{d^2y}{dt^2} = f\left(t, y(t), \frac{dy}{dt}(t)\right)$ → 2nd order ODE

• Linear / nonlinear

2nd order general ODE

$$a \frac{d^2y}{dt^2} + b \frac{dy}{dt} + cy = d$$

(i) if a, b, c do not depend on $y, \frac{dy}{dt}, \frac{d^2y}{dt^2}$

then the equation is Linear ODE

(ii) if any of a, b, c depend on at least one of $y, \frac{dy}{dt}, \frac{d^2y}{dt^2}$

then the equation is nonlinear ODE.

↳ $c = y$

• Conditions necessary to get complete solution

(i) IVP (Initial value problem)

if $\frac{d^ny}{dt^n} = f\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}\right)$

then if I have

$$y(0) = y_0^{(0)}$$

$$\frac{dy}{dt}(0) = y_0^{(1)}$$

$$\frac{d^2y}{dt^2}(0) = y_0^{(2)}$$

$$\vdots$$

$$\frac{d^{n-1}y}{dt^{n-1}}(0) = y_0^{(n-1)}$$

for n^{th} order ODE we need n conditions

(2) BVP (Boundary Value problem)

$$\frac{d^n y}{dt^n} = f\left(t, y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}\right)$$

$n=2 \longrightarrow \quad y(0) = y_0, \quad y(t=L) = y_L$

$n=4 \longrightarrow \quad y(0) = y_0^{(0)}, \quad y(t=L) = y_L^{(0)}$

$\frac{dy}{dt}(0) = y_0^{(1)}, \quad \frac{dy}{dt}(t=L) = y_L^{(1)}$