Lecture 34

- Approximation of differentials

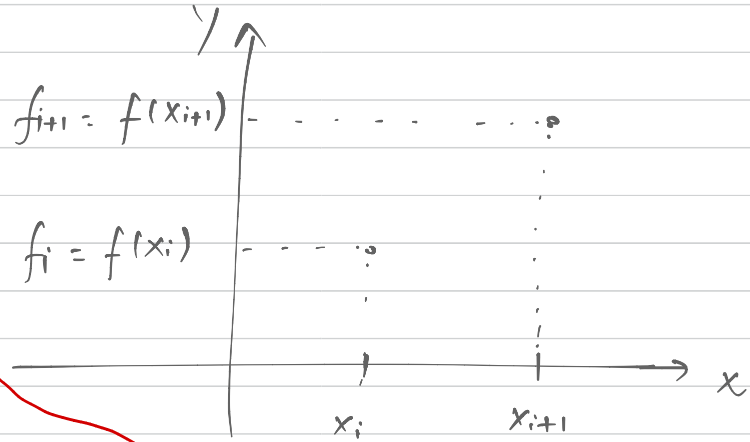
notation:

- $\frac{df}{dx}(x) = f'(x) = \dot{f}(x) = f^{(1)}(x)$

- $f''(x) = \frac{d^2 f}{dx^2}(x) = \ddot{f}(x) = f^{(2)}(x)$

- $f'''(x) = \frac{d^3 f}{dx^3}(x) = \dddot{f}(x) = f^{(3)}(x)$

- $f^{(4)}(x) = \frac{d^4 f}{dx^4}(x)$

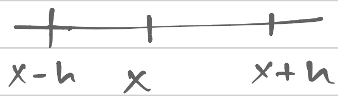


Problem Approximate $f'(x_i)$ and $f'(x_{i+1})$

using $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$

Start with definition (assume $h > 0$)

$$f'(x) = \frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



forward difference

$$= \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

backward difference

Taking out " $\lim_{h \to 0}$ "

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}, \quad h = x_{i+1} - x_i$$

$\downarrow$

$f(x_i + h) = f(x_{i+1})$

$$f'(x_{i+1}) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Apply Taylor's series to

(i) understand error due to forward/backward difference approximation

(ii) develop more accurate approximations

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)(x_{i+1}-x_i)^2}{2!} + \frac{f'''(x_i)(x_{i+1}-x_i)^3}{3!} + \frac{f^{iv}(x_i)(x_{i+1}-x_i)^4}{4!} + \ldots$$

define $h = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + \frac{f^{iv}(x_i)}{4!}h^4 + \ldots$$

(1) $f(x_{i+1}) = f(x_i) + O(h)$

(2) $f(x_{i+1}) = f(x_i) + f'(x_i)h + O(h^2)$

(3) $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + O(h^3)$

(4) $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + O(h^4)$

$$A = (a_1 + a_2 h + a_3 h^2 + \dots)(b_1 + b_2 \epsilon + b_3 \epsilon^2 + \dots)$$

$$= a_1 b_1 + a_2 b_1 h + a_1 b_2 \epsilon + a_2 b_2 \epsilon h + \dots$$

$$A = a_1 b_1 + O(\epsilon h)$$

$$A = a_1 b_1 + a_2 b_1 h + a_1 b_2 \epsilon + a_2 b_2 \epsilon h + O(\epsilon^2 h^2)$$

- $\frac{1}{h} O(h^n) = O(h^{n-1})$
- $h\, O(h^n) = O(h^{n+1})$
- $(-1)\, O(h^n) = O(h^n)$
- if α is some fixed number then $\alpha\, O(h^n) = O(h^n)$

Eq^n (1)

$$f(x_{i+1}) = f(x_i) + O(h)$$

$$O(h) = f'(x_i) h + \frac{f''(x_i)}{2!} h^2 + \frac{f'''(x_i)}{3!} h^3 + \dots$$

$$\frac{1}{h} O(h) = f'(x_i) + \frac{f''(x_i)}{2!} h + \dots$$

$$= O(h^0) = O(1)$$

$$h\, O(h) = f'(x_i) h^2 + \frac{f''(x_i)}{2!} h^3 + \dots$$

$$= O(h^2)$$

Eqⁿ(2) $$f(x_{i+1}) = f(x_i) + f'(x_i)\,h + O(h^2)$$

$$\Rightarrow f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{1}{h}O(h^2)$$

$$\Rightarrow \boxed{f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)}$$

$$\boxed{f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}}$$

error if I approximate $f'(x_i)$ by $\frac{f(x_{i+1}) - f(x_i)}{h}$

Eq (3) $$f(x_{i+1}) = f(x_i) + f'(x_i)\,h + \frac{f''(x_i)}{2}h^2 + O(h^3)$$

$$\Rightarrow f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)}{2}h^2 - \frac{1}{h}O(h^3)$$

$$\Rightarrow f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)}{2}h^2 + O(h^2)$$

Now consider three data points

$(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, $(x_{i+2}, f(x_{i+2}))$

& assume $x_{i+1} - x_i = x_{i+2} - x_{i+1} = h$

$$f''(x_i) \approx \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$$