Lecture 29

Consider $n+1$ data

$$(x_1,\ y_1 = f(x_1)),\quad (x_2,\ y_2 = f(x_2)),\quad \dots,\quad (x_{n+1},\ y_{n+1} = f(x_{n+1}))$$

Lagrange

$$\hat{y} = \hat{f}(x) = \hat{z}(x)\, a \quad \rightarrow \quad n^{th} \text{ order polynomial}$$

where

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{bmatrix}, \qquad \hat{z} = [L_1(x), L_2(x), \dots, L_{n+1}(x)]$$

$$\hat{J} a = b$$

$\hat{J} = I$ identity matrix

$$a = b = \begin{bmatrix} y_1 \\ y_2 \\ | \\ y_{n+1} \end{bmatrix}$$

& for any $i = 1, 2, \dots, n+1$

$$L_i(x) = \frac{\sum\limits_{\substack{j=1 \\ j\neq i}}^{n+1} (x - x_j)}{\sum\limits_{\substack{j=1 \\ j\neq i}}^{n+1} (x_i - x_j)}$$

Newton's interpolation

$$\tilde{y} = \tilde{f}(x) = \tilde{z}(x)\, d$$

$$d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n+1} \end{bmatrix}, \qquad \tilde{z}(x) = [1,\ x - x_1,\ (x-x_1)(x-x_2),\ \dots,\ (x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)]$$

$$\tilde{J} d = b \Rightarrow d = \begin{bmatrix} y[1] \\ y[2,1] \\ \vdots \\ y[n+1, n, \dots, 2, 1] \end{bmatrix}$$

Example of linear interpolation $\tilde{y} = d_1 + d_2(x - x_1)$

$$d_1 = y_1, \quad d_2 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\tilde{y}(x_1) = d_1 = y_1$$

$$\tilde{y}(x_2) = d_1 + d_2(x_2 - x_1) = y_2$$

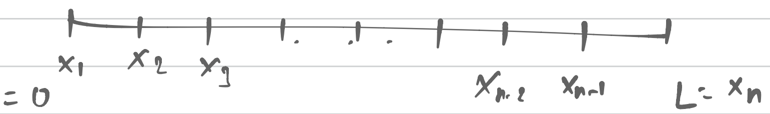
Property of Lagrange's interpolation

$a_i \longrightarrow$ Values of functions at x_i

How this property helps us

$$K \frac{d^2 T}{dx^2} = q$$

$\hat{T} = \hat{T}(x) =$ approximate temperature field



Choice 1 $\quad \hat{T}(x) = [L_1(x), L_2(x), \ldots, L_n(x)] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

Choice 2 $\quad \hat{T}(x) = [1, x - x_1, (x-x_1)(x-x_2), \ldots, (x-x_1)(x-x_2)\ldots(x-x_n)] \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$

$$J a = b$$

for choice 1, some J, $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, some b

force choice 2, some J, $a = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$, some b

Graphing Lagrange basis

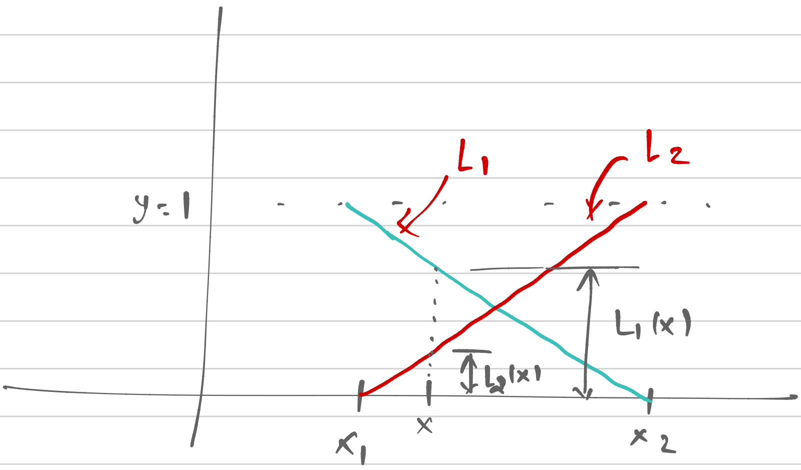
(i) Linear interpolation

(x_1, y_1), (x_2, y_2)

$$\hat{f}(x) = y_1 \boxed{\frac{(x - x_2)}{(x_1 - x_2)}} + y_2 \boxed{\frac{(x - x_1)}{(x_2 - x_1)}}$$

a_1, a_2

L_1, L_2



$L_1(x_1) = 1$

$L_1(x_2) = 0$

$L_2(x_1) = 0$

$L_2(x_2) = 1$

take any point $x \in [x_1, x_2]$

$$L_1(x) + L_2(x) = 1$$

$$y = f(x) = c \Rightarrow y_1 = c, \quad y_2 = c$$

then $\hat{f}(x) = c \quad \forall x \in [x_1, x_2]$

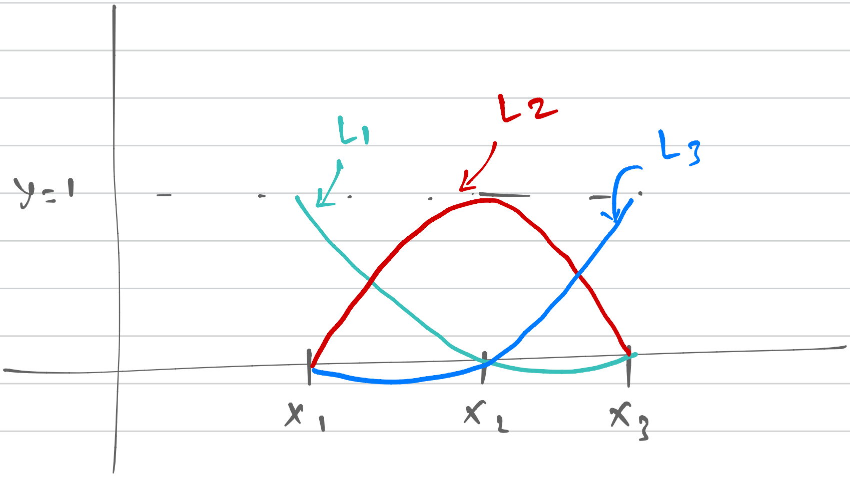
$$y_1 L_1(x) + y_2 L_2(x) = c$$

$\Rightarrow c\left(L_1(x) + L_2(x)\right) = c$ true only if $L_1(x) + L_2(x) = 1$ for any $x \in [x_1, x_2]$

Quadratic interpolation $(x_1, y_1), (x_2, y_2), (x_3, y_3)$

$$\hat{f}(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

$$L_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \quad ; \; \ldots$$

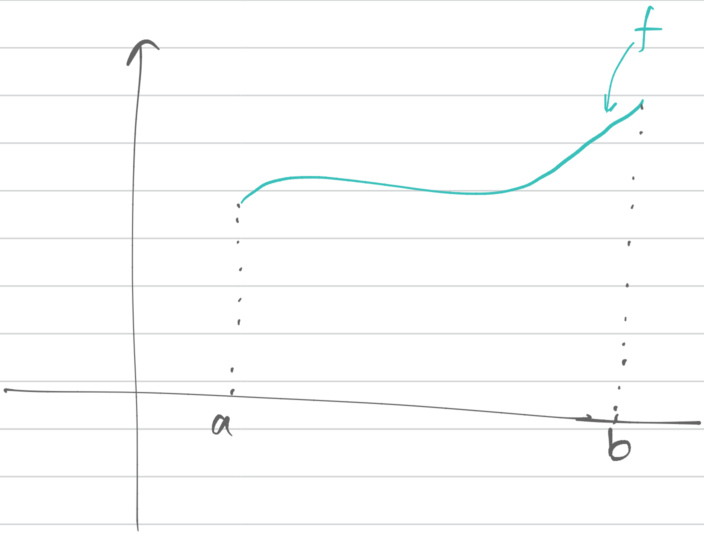


$L_1(x_1) = 1$	$L_2(x_1) = 0$
$L_1(x_2) = 0$	$L_2(x_2) = 1$
$L_1(x_3) = 0$	$L_2(x_3) = 0$

$$L_1(x) + L_2(x) + L_3(x) = 1 \quad \text{for any } x \in [x_1, x_3]$$

Approximation of Integrals

$f : [a,b] \rightarrow (-\infty, \infty)$



$$I[f] = \int_a^b f(x)\,dx$$

take n discrete points in $[a,b]$

such that

$$x_1 = a < x_2 < x_3 \; \cdots \; < x_{n-1} < x_n = b$$

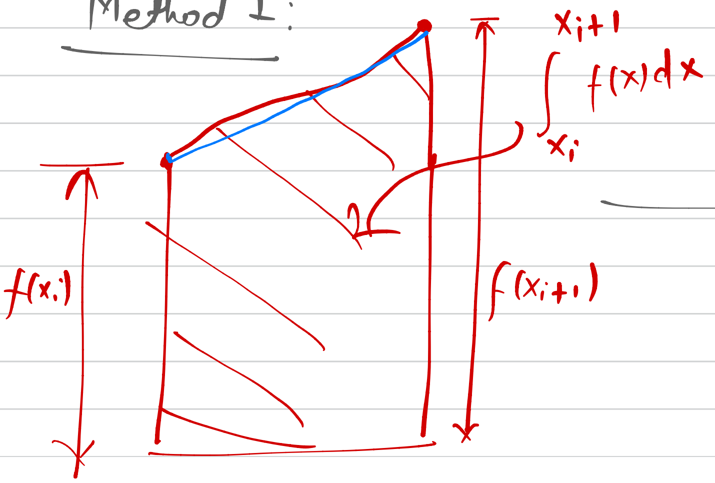
uniform discretization $\quad x_1 = a, \quad x_2 = a+h, \quad x_3 = a+2h,$

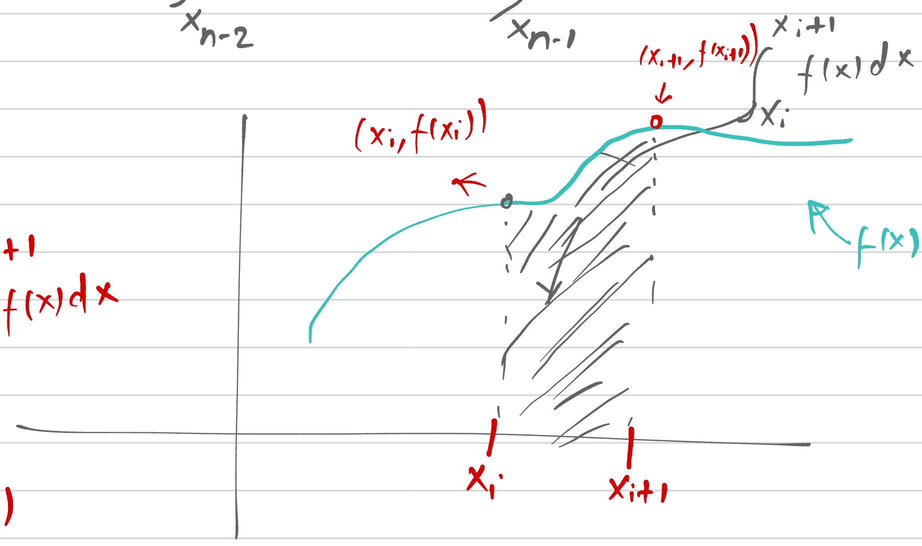
$\cdots, \; x_{n-1} = a+(n-2)h$

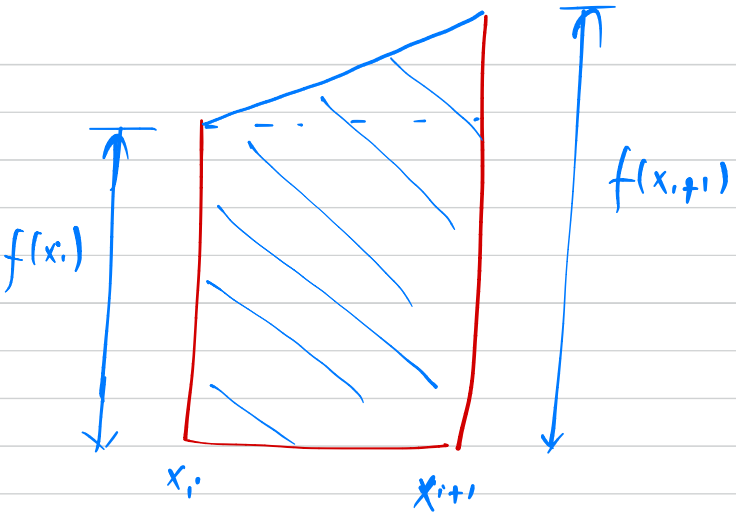
$x_n = a+nh = b$

$$I[f] = \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx + \cdots + \int_{x_i}^{x_{i+1}} f(x)dx + \cdots + \int_{x_{n-2}}^{x_{n-1}} f(x)\,dx + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

Method 1:







$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{\left(f(x_{i+1}) + f(x_i)\right)(x_{i+1} - x_i)}{2}$$

Some i

$$I[f] = \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$\approx \frac{(x_2 - x_1)}{2}\left(f(x_2) + f(x_1)\right) + \frac{(x_3 - x_2)}{2}\left(f(x_3) + f(x_2)\right) + \cdots + \frac{(x_n - x_{n-1})}{2}\left(f(x_n) + f(x_{n-1})\right)$$

if you have uniform discretization $\quad (a + (n-1)h = b)$

$$x_i = a + (i-1)h$$

$$= h\left[\frac{f(a)}{2} + f(a+h) + f(a+2h) + \cdots + f(a+(n-2)h) + \frac{f(a+(n-1)h)}{2}\right]$$

Method 2 : I want to approximate

$$\int_{x_i}^{x_{i+1}} f(x)\,dx, \quad \boxed{(x_i, f(x_i)),\ (x_{i+1}, f(x_{i+1}))}$$

$$\approx \int_{x_i}^{x_{i+1}} \hat{f}(x)\,dx$$

$$\hat{f}(x) = f(x_i)\left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right) + f(x_{i+1})\left(\frac{x - x_i}{x_{i+1} - x_i}\right)$$

$$= \frac{(x_{i+1} - x_i)}{2} \left(f(x_i) + f(x_{i+1}) \right)$$