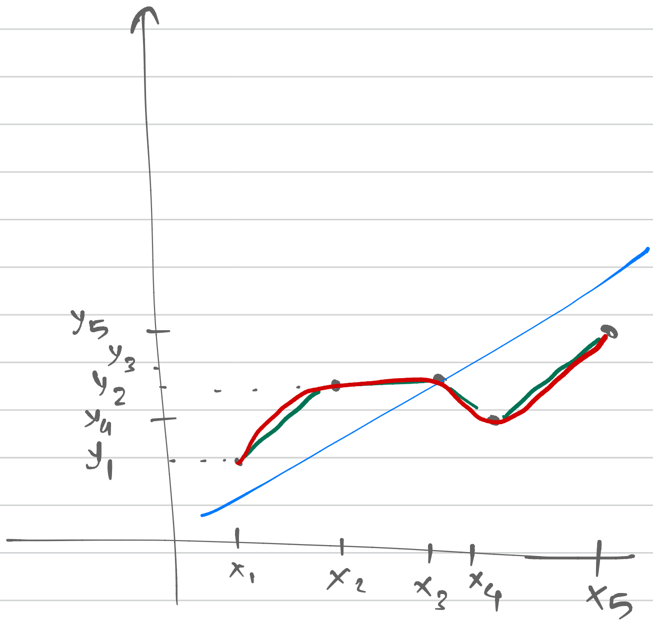
Lecture 27

· Interpolation

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

find a curve $f = f(x)$
such that



$f(x_1) = y_1$

$f(x_2) = y_2$

$\vdots$

$f(x_n) = y_n$

Idea take f function as polynomial in X.

$z(x)\, a \quad = f(x) = a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1}$
$= z(x)\, a$

$z = [1, x, x^2, \ldots, x^{n-1}]$

Find unknowns $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ s.t

(1) $f(x_1) = y_1 = a_1 + a_2 x_1 + a_3 x_1^2 + \cdots + a_n (x_1)^{n-1}$

(2) $f(x_2) = y_2 = a_1 + a_2 x_2 + a_3 x_2^2 + \cdots + a_n (x_2)^{n-1}$

$\vdots$

(n) $f(x_n) = y_n = a_1 + a_2 x_n + a_3 x_n^2 + \cdots + a_n (x_n)^{n-1}$

$$Ja = b$$

$$J = \begin{bmatrix} — & z(x_1) & — \\ — & z(x_2) & — \\ & \vdots & \\ — & z(x_n) & — \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

$$\begin{array}{ccccccc} x_1 & < & x_2 & < \cdots & x_3 & < & x_n \\ =1 & & =2 & & =5 & & =10 \end{array}$$

$n = 4$

$Ax = b$

error in x $\leq$ cond[A] * error in b

$$J = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 5 & 25 & 125 \\ 1 & 10 & 100 & 1000 \end{bmatrix}$$

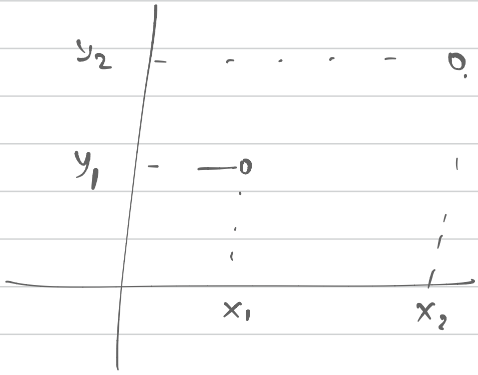
• Newton's interpolation method

$$f(x) = a_1 + a_2 (x - x_1)$$

" direct method

$$f(x) = a_1 + a_2 x \quad ''$$

$$Z(x) = [1, \; x - x_1]$$



- $f(x_1) = y_1 = a_1 + a_2(x_1 - x_1) = a_1 \Rightarrow a_1 = y_1$

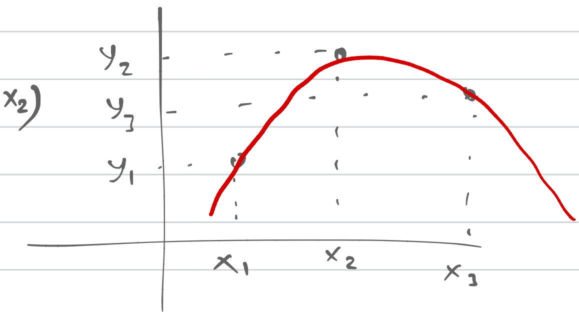
- $f(x_2) = y_2 = a_1 + a_2(x_2 - x_1) = y_1 + a_2(x_2 - x_1)$

$$\Rightarrow a_2 = \frac{y_2 - y_1}{x_2 - x_1}$$

Example of quadratic polynomial

$$f(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2)$$

$$\text{"} \; f(x) = a_1 + a_2 x + a_3 x^2 \; \text{''}$$



- $f(x_1) = y_1 \Rightarrow y_1 = a_1$
- $f(x_2) = y_2 \Rightarrow y_2 = a_1 + a_2(x_2 - x_1) \Rightarrow a_2 = \dfrac{y_2 - y_1}{x_2 - x_1}$
- $f(x_3) = y_3 \Rightarrow y_3 = a_1 + a_2(x_3 - x_1) + a_3(x_3 - x_1)(x_3 - x_2)$

$$\Rightarrow y_3 - y_1 - \frac{(y_2 - y_1)}{(x_2 - x_1)}(x_3 - x_1) = a_3 (x_3 - x_1)(x_3 - x_2)$$

$$\Rightarrow \boxed{a_3 = \frac{\dfrac{(y_3 - y_1)}{(x_3 - x_1)} - \dfrac{(y_2 - y_1)}{(x_2 - x_1)}}{x_3 - x_2}}$$

$$a_3 (x_3 - x_1)(x_3 - x_2) = y_3 - y_1 - \frac{(y_2 - y_1)}{(x_2 - x_1)} (x_3 - x_1)$$

$$= y_3 - y_2 + (y_2 - y_1) - \frac{(y_2 - y_1)}{(x_2 - x_1)} (x_3 - x_1)$$

$$= (y_3 - y_2) - (y_2 - y_1) \left[\frac{x_3 - x_1}{x_2 - x_1} - 1 \right]$$

$$\Rightarrow a_3 (x_3 - x_1)(x_3 - x_2) = (y_3 - y_2) - (y_2 - y_1) \left[\frac{x_3 - x_2}{x_2 - x_1} \right]$$

$$\Rightarrow a_3 = \frac{(y_3 - y_2)}{(x_3 - x_1)(x_3 - x_2)} - \frac{(y_2 - y_1)}{(x_3 - x_1)\cancel{(x_3 - x_2)}} \frac{\cancel{(x_3 - x_2)}}{(x_2 - x_1)}$$

$$= \frac{y_3 - y_2}{(x_3 - x_1)(x_3 - x_2)} - \frac{y_2 - y_1}{(x_3 - x_1)(x_2 - x_1)}$$

$$\boxed{a_3 = \frac{\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1}}{x_3 - x_1}}$$

$$f(x) = a_1 + a_2 (x - x_1) + a_3 (x - x_1)(x - x_2)$$

$$= \underbrace{[1, \; x - x_1, \; (x - x_1)(x - x_2)]}_{z(x)} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{a}$$

- $f(x_1) = y_1, \quad f(x_2) = y_2, \quad f(x_3) = y_3$

$$Ja = b, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$J = \begin{bmatrix} \text{---} & Z(x_1) & \text{---} \\ \text{---} & Z(x_2) & \text{---} \\ \text{---} & Z(x_3) & \text{---} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & x_2 - x_1 & 0 \\ 1 & x_3 - x_1 & (x_3 - x_1)(x_3 - x_2) \end{bmatrix}$$

- finite divided differences

$y_1, \; y_2, \; y_3, \; \dots, \; y_n$

$x_1, \; x_2, \; x_3, \; \dots, \; x_n$

- $y[i] = y_i$

- $y[j,i] = \dfrac{y_j - y_i}{x_j - x_i} \quad \rightarrow \quad y[2,1] = \dfrac{y_2 - y_1}{x_2 - x_1}$

- $y[k,j,i] = \dfrac{y[k,j] - y[j,i]}{x_k - x_i} \quad \rightarrow \quad y[3,2,1] = \dfrac{y[3,2] - y[2,1]}{x_3 - x_1}$

$$= \frac{\dfrac{y_3 - y_2}{x_3 - x_2} - \dfrac{y_2 - y_1}{x_2 - x_1}}{x_3 - x_1}$$

- $$y[m,k,j,i] = \frac{y[m,k,j] - y[k,j,i]}{x_m - x_i}$$

$\vdots$

- $$y[n, n-1, n-2, \ldots, 3, 2, 1] = \frac{y[n, n-1, n-2, \ldots, 3, 2] - y[n-1, n-2, \ldots, 2, 1]}{x_n - x_1}$$

- line

$$a_1 = y[1] = y_1$$

$$a_2 = y[2,1] = \frac{y_2 - y_1}{x_2 - x_1}$$

- quadratic

$$a_1 = y[1] = y_1$$

$$a_2 = y[2,1] = \frac{y_2 - y_1}{x_2 - x_1}$$

$$a_3 = y[3,2,1] = \frac{y[3,2] - y[2,1]}{x_3 - x_1}$$

- $(n-1)^{th}$ order polynomial $\longrightarrow (x_1, y_1), \ldots, (x_n, y_n)$

$$f(x) = a_1 + a_2(x-x_1) + a_3(x-x_1)(x-x_2) + \cdots + a_n(x-x_1)(x-x_2)\cdots(x-x_{n-1})$$

$a_1 = y[1] = y_1$

$a_2 = y[2,1]$

$a_3 = y[3,2,1]$

$a_4 = y[4,3,2,1]$

$\vdots$

$$a_n = y[n, n-1, n-2, \ldots, 3, 2, 1]$$

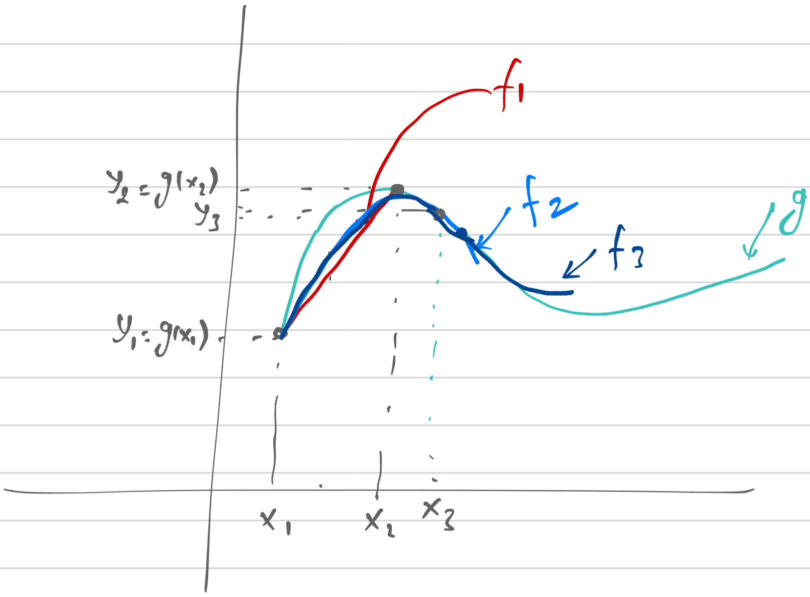
<u>Suppose</u> $\quad y = g(x)$

$$(x_1, y_1), (x_2, y_2) \ldots, (x_n, y_n)$$

$$(x_1, g(x_1)), (x_2, g(x_2)), \ldots, (x_n, g(x_n))$$

$(x_1, y_1), (x_2, y_2)$

$$f_1(x) = a_1 + a_2(x - x_1)$$



$(x_1, y_1), (x_2, y_2), (x_3, y_3)$

$$f_2(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2)$$

$$= f_1(x) + a_3(x - x_1)(x - x_2)$$

$$a_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{g(x_2) - g(x_1)}{x_2 - x_1} \approx \frac{dg}{dx}(x_1)$$

$$a_3 = \frac{\dfrac{y_3 - y_2}{x_3 - x_2} - \dfrac{y_2 - y_1}{x_2 - x_1}}{x_3 - x_1}$$

$$\approx \frac{\dfrac{dg}{dx}(x_3) - \dfrac{dg}{dx}(x_1)}{x_3 - x_1} \approx \frac{d^2 g}{dx^2}(x_1)$$

Lecture 28

· errors in polynomial interpolation

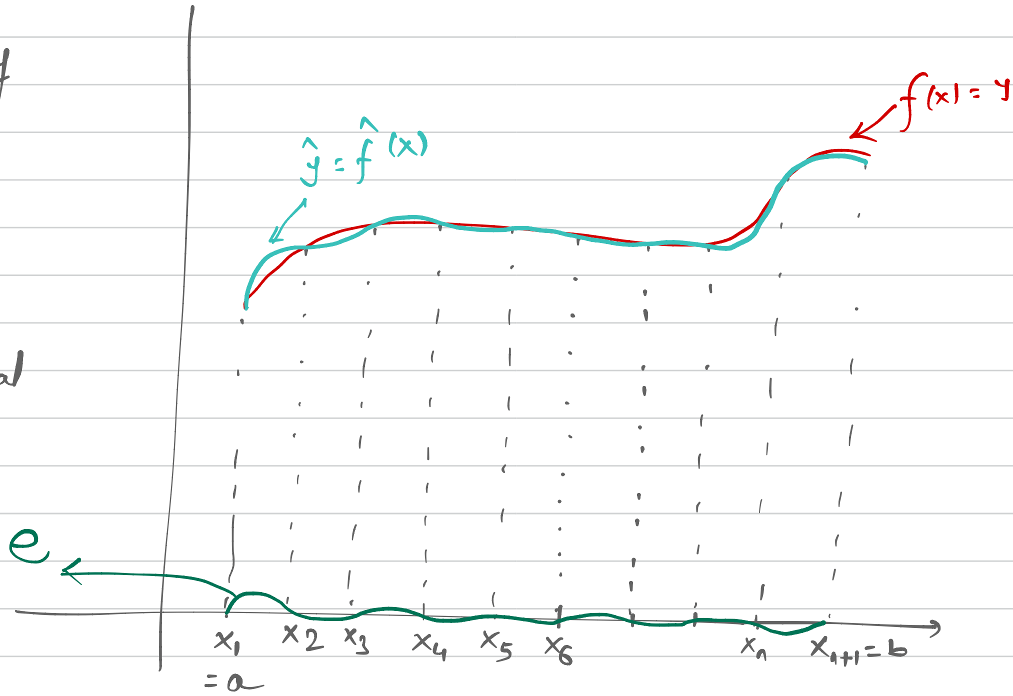
let us assume data comes from function $y = f(x)$

n+1 data points $(x_1, y_1 = f(x_1)), (x_2, y_2 = f(x_2)), \dots, (x_{n+1}, y_{n+1} = f(x_{n+1}))$

We fit a polynomial of n^{th} order:

$\hat{y} = \hat{f}(x)$

↑ n^{th} order polynomial



Error: $e = e(x) = f(x) - \hat{f}(x)$

$e(x_1) = 0$

$e(x_2) = 0$

⋮

$e(x_{n+1}) = 0$

has $(n+1)$ roots and these roots are $x_1, x_2, \dots, x_{n+1}$

$\Downarrow$

$$e(x) \approx H(x - x_1)(x - x_2) \dots (x - x_{n+1})$$

$$\Rightarrow f(x) - \hat{f}(x) = H(x - x_1)(x - x_2) \dots (x - x_{n+1})$$

$$\underbrace{\frac{d^{n+1}}{dx^{n+1}} f(x) - \cancel{\frac{d^{n+1}}{dx^{n+1}} \hat{f}(x)}^{\,0}}_{} = H \underbrace{\cancel{\frac{d^{n+1}}{dx^{n+1}} \left[(x-x_1) \cdots (x-x_{n+1}) \right]}^{\,(n+1)!}}_{}$$

$$H = \frac{1}{(n+1)!} \frac{d^{n+1}}{dx^{n+1}} f(x)$$

$$e \simeq \frac{1}{(n+1)!} \left[\frac{d^{n+1}}{dx^{n+1}} f(x) \right] (x-x_1) \cdots (x-x_{n+1})$$

$$\frac{d^{n+1}}{dx^{n+1}} \left[x^{n+1} + \alpha_1 x^n + \alpha_2 x^{n-1} + \cdots + \alpha_n \cdot 1 \right]$$

$$= \frac{d^{n+1}}{dx^{n+1}} x^{n+1}$$

$$\frac{d}{dx} x^{n+1} = (n+1) x^n$$

$$\frac{d^2}{dx^2} x^{n+1} = (n+1) n x^{n-1}$$

$$\vdots$$

$$\frac{d^{n+1}}{dx^{n+1}} x^{n+1} = (n+1) n (n-1) \cdots 3 \cdot 2 \cdot 1$$

$$= (n+1)!$$

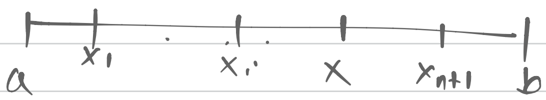
assume that

$x_i \in [a, b]$, $\quad i = 1, 2, \ldots, n+1$

for any $x \in [a, b]$, and $i = 1, 2, \ldots, n+1$

I have following

$$|x - x_i| \leq |b - a|$$



$$|(x-x_1)(x-x_2) \cdots (x-x_{n+1})| \leq |b-a|^{n+1}$$

$$\leq |x-x_1| \, |x-x_2| \, |x-x_3| \cdots |x-x_{n+1}|$$

$$\leq |b-a|^{n+1}$$

Assume that there is a number M such that for any $x \in [a, b]$,

$$\boxed{\left| \frac{d^{n+1}}{dx^{n+1}} f(x) \right| \leq M}$$

Then

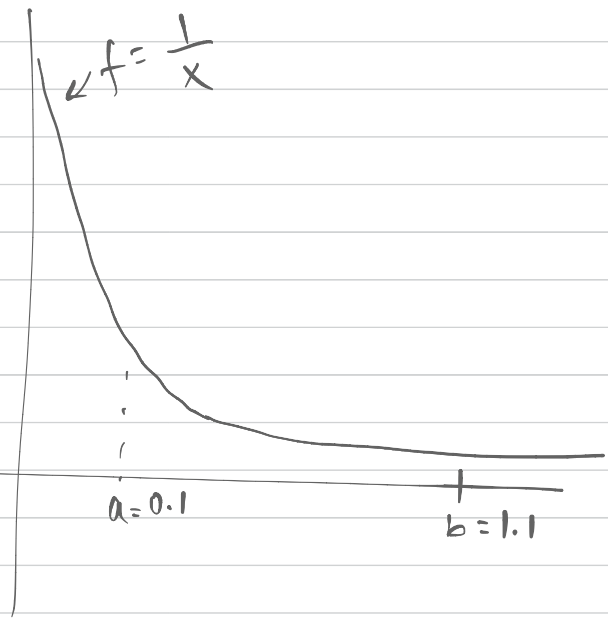
$$\boxed{|e(x)| \leq \frac{M}{(n+1)!} |b-a|^{n+1}} \quad \text{for all } x \in [a, b]$$

Suppose $\left| \frac{d^{(i)}}{dx^i} f(x) \right| \leq M$ for any integer i

$$|e(x)| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

Counter example

$f = \frac{1}{x}$, $\quad x \in [\underset{a}{0.1}, \underset{b}{1.1}]$



$$\frac{df}{dx} = \frac{(-1)}{x^2}, \quad \frac{d^2 f}{dx^2} = \frac{(-1)^2 (1)(2)}{x^3}$$

$$\frac{d^i f}{dx^i} = \frac{(-1)^i (i)(i-1)\ldots(2)(1)}{x^{i+1}}$$

$$= \frac{(-1)^i \, i!}{x^{i+1}}$$

$0.1 \leq x \leq 1$

$\frac{1}{x} \leq \frac{1}{0.1}$

$$\left| \frac{d^i f}{dx^i} \right| \leq \frac{i!}{x^{i+1}} \leq \frac{i!}{(0.1)^{i+1}}$$

$x \in [0.1, 1]$

$$\left| \frac{d^i f}{dx^i} \right| \leq (10)^{i+1} \; i! \qquad \underline{\text{any } i}$$

$$|e| \leq \frac{M}{(n+1)!} |b-a|^{n+1}, \qquad M \geq \left| \frac{d^{n+1} f}{dx^{n+1}} \right|$$

$$M = (10)^{n+2} (n+2)!$$

$$\leq \frac{(10)^{n+2} (n+2)!}{(n+1)!} \cancel{(1)^{n+1}}$$

$$\boxed{|e| \leq (10)^{n+2} (n+2)}$$

- Lagrange interpolation method

Consider $(n+1)$ data $(x_1, y_1), \ldots, (x_{n+1}, y_{n+1})$

&

Consider n^{th} order polynomial

$$\hat{y} = \hat{f}(x) = z(x)\, a, \qquad z(x) = [z_1(x), z_2(x), \ldots, z_{n+1}(x)]$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{bmatrix}$$

Direct method

$$Z = [1, x, x^2, \ldots, x^{n+1}] \longrightarrow J = \begin{bmatrix} 1 & x_1 & x_1^2 \cdots & x_1^{n+1} \\ 1 & x_2 & x_2 \cdots & x_2^{n+1} \\ \cdot & & - & \cdot \end{bmatrix}$$

Newton's interpolation

$$Z = [1, x - x_1, (x-x_1)(x-x_2), \ldots, (x-x_1)(x-x_2)\cdots(x-x_n)]$$

$$\longrightarrow \quad J = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & \cdots & 0 \\ 1 & x_n - x_1 & (x_n - x_1)(x_n - x_2) & \end{bmatrix}$$

- Lagrange method

$$J = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{bmatrix}$$

- <u>Line example</u> $\quad (x_1, y_1), \ (x_2, y_2)$

$$\hat{y} = \hat{f}(x) = \left(\frac{x - x_2}{x_1 - x_2}\right) a_1 + \left(\frac{x - x_1}{x_2 - x_1}\right) a_2 = z(x) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$z(x) = \left[\frac{x - x_2}{x_1 - x_2}, \ \frac{x - x_1}{x_2 - x_1} \right]$$

$$\hat{f}(x_1) = y_1 \ \rightarrow \ \cancelto{1}{\left(\frac{x_1 - x_2}{x_1 - x_2}\right)} a_1 + \cancelto{0}{\left(\frac{x_1 - x_1}{x_2 - x_1}\right)} a_2 = y_1$$

$$\rightarrow \quad a_1 = y_1$$

$$\hat{f}(x_2) = y_2 \Rightarrow \left(\cancel{\frac{x_2 - x_2}{x_1 - x_2}}^{\,0}\right) a_1 + \left(\cancel{\frac{x_2 - x_1}{x_2 - x_1}}^{\,1}\right) a_2 = y_2$$

$$\Rightarrow \qquad a_2 = y_2$$

$$Ja = b, \qquad b = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$J = \begin{bmatrix} \text{---} \; z(x_1) \; \text{---} \\ \text{---} \; z(x_2) \; \text{---} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- <u>Quadratic</u>