Eigenvalues and eigenvectors

Def: A pair of number λ and column vector $x_{n\times 1}$ are called eigenvalue and eigenvector of square matrix $A_{n\times n}$ if

$$Ax = \lambda x \quad \text{———} \quad (1)$$

Remark 1: Any matrix $A_{n\times n}$ can have multiple pairs (λ, x) that satisfy (1). I.e. (λ_1, x_1) s.t. $Ax_1 = \lambda_1 x_1$, (λ_2, x_2) s.t. $Ax_2 = \lambda_2 x_2$, ... and (λ_n, x_n) s.t. $Ax_n = \lambda_n x_n$.

Remark 2: Any matrix $A_{n\times n}$ can have atmost n pairs of (λ, x).

Remark 3: Any matrix $A_{n\times n}$ can have atmost n unique eigenvalues, i.e. $\lambda_i \neq \lambda_j$ for all $i \neq j$

Remark 4: If a matrix $A_{n\times n}$ has n unique eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, then eigenvectors $x_1, x_2, \ldots, x_n$ are linearly independent. Collection of any m vectors, $a_1, a_2, \ldots, a_m$ are called linearly independent if

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0 \quad \text{only if} \quad \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0.$$

What does (1) means

Given a vector x (size n) and matrix A $(n\times n)$, then Ax is another vector y (size n), i.e.

$$y = Ax.$$

So when we multiply a vector with matrix, we get a new vector.

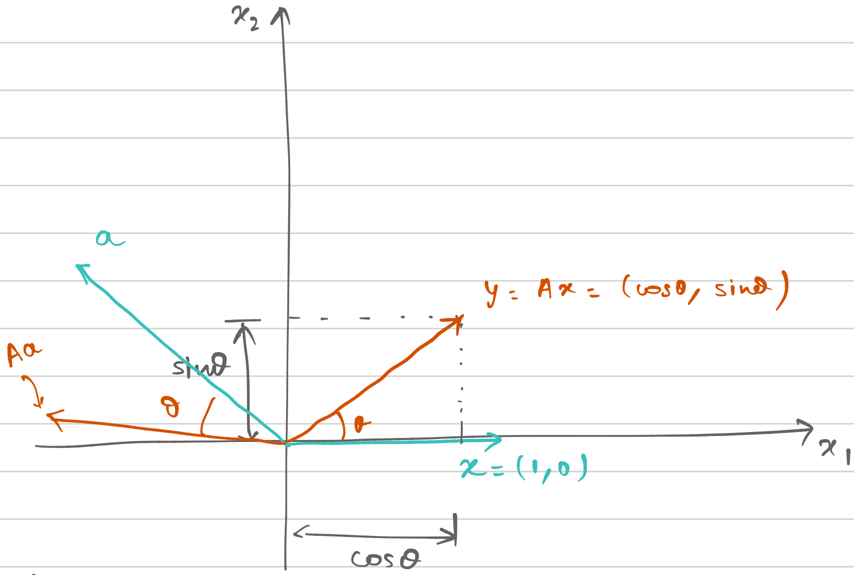
<u>Action of A on x</u> is that A <u>rotates</u> and <u>scales</u> x to give a new vector y.

<u>In $n=2$ setting</u>, consider examples

(i) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

<u>Then</u>

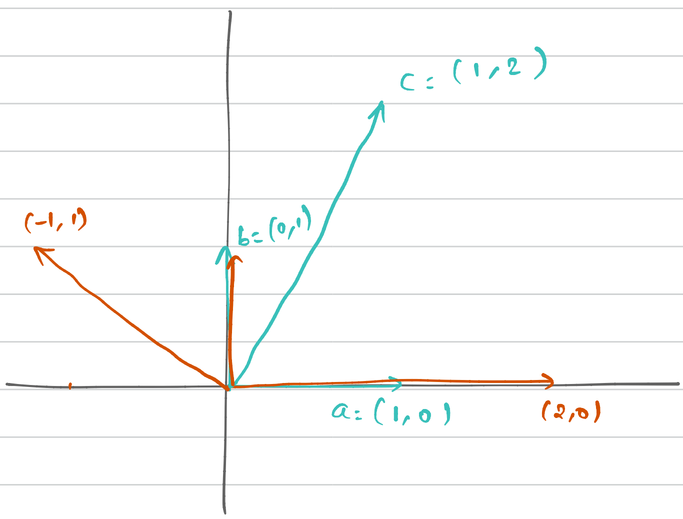
$$\underbrace{A\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{vector } x} = \underbrace{\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}_{\text{new vector } y}$$



So matrix A when applied to vector x simply rotates vector in anticlockwise direction by angle θ.

(ii) $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix}$$



<u>So</u> $\underset{a}{A\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\underset{b}{A\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\underset{c}{A\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The two examples show that in general Ax is new vector that is obtained as a result of rotation and scaling of x whereas eigenvectors and eigenvalues are special pair (λ, x) such that Ax is a vector that is not rotated but only scaled by A.

For example $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, eigenvalues are complex numbers!

$$\lambda = \cos\theta \pm i \sin\theta$$

For example $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, eigenvalues are real and unique!

$\lambda_1 = \frac{3+\sqrt{5}}{2}$, $x_1 = \frac{1}{\sqrt{\lambda_1^2 - 2\lambda_1 + 5}} \begin{bmatrix} 1-\lambda_1 \\ 1 \end{bmatrix}$ pair 1

$\lambda_2 = \frac{3-\sqrt{5}}{2}$, $x_2 = \frac{1}{\sqrt{\lambda_2^2 - 2\lambda_2 + 5}} \begin{bmatrix} 1-\lambda_2 \\ 1 \end{bmatrix}$ pair 2

$\underline{1}$, $Ax_1 = \lambda_1 x_1$ (no rotation)

$Ax_2 = \lambda_2 x_2$ (no rotation)

How does eigenvalues and eigenvectors help us

(A) Let $A_{n\times n}$ has n unique eigen pairs $(\lambda_1, \underline{x}_1), (\lambda_2, \underline{x}_2), \ldots, (\lambda_n, \underline{x}_n)$

— all $\underline{x}_i$ are $n\times 1$ column vectors (eigen vectors)!

— Consider vector $\underline{a} = \alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \cdots + \alpha_n \underline{x}_n$

<u>where</u> $\alpha_1, \alpha_2, \ldots, \alpha_n$ are some numbers.

<u>Then</u>

$$A\underline{a} = A(\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \cdots + \alpha_n \underline{x}_n)$$

(since $A(x+y) = Ax + Ay$ and $A(\alpha x) = \alpha Ax$)

$$= \alpha_1 A\underline{x}_1 + \alpha_2 A\underline{x}_2 + \cdots + \alpha_n A\underline{x}_n$$

$$\Rightarrow \boxed{A\underline{a} = \alpha_1 \lambda_1 \underline{x}_1 + \alpha_2 \lambda_2 \underline{x}_2 + \cdots + \alpha_n \lambda_n \underline{x}_n}$$

<u>That is</u>

if we can represent any vector $\underline{a}$ using eigenvectors $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$, then we can very easily compute $A\underline{a}$ by above formula.

<u>Unique</u> and <u>real</u> (ie. not complex numbers) n eigenvalues and corresponding eigen vectors provide a <u>means to</u> represent <u>any column vector</u> of n elements using

$$\underline{y} = \alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \cdots + \alpha_n \underline{x}_n$$

and this representation allows calculation of Ay trivially as

$$Ay = \alpha_1 \lambda_1 \underline{x}_1 + \alpha_2 \lambda_2 \underline{x}_2 + \cdots + \alpha_n \lambda_n \underline{x}_2$$

Remark 5: For each eigenvalue λ, there are infinitely many eigenvectors. Because, if $\underline{x}$ is a eigenvector, that is,

$$Ax = \lambda x$$

then αx is also eigenvector for any number α.

Check: let $y = \alpha x$

$$A(\alpha x) = \alpha Ax = \alpha \lambda x = \lambda (\alpha x)$$

$$\Rightarrow \quad Ay = \lambda y$$

So $y = \alpha x$ is also eigenvector.

Therefore, for each eigen value λ, we have a family or collection of eigenvectors and we typically consider only one member of this family.

We will come to utility of eigenvalues and eigenvectors, but, let us first discuss how we can compute λ and x.

How to solve eigen value problem $Ax = \lambda x$?

We have $\quad Ax = \lambda x$ $\quad$ let I is identity matrix

meaning, for any $\underline{x}$

$$Ix = x$$

$$\Rightarrow Ax = \lambda I x$$

$$(2) \quad \Rightarrow (A - \lambda I)x = 0$$

Equation (2) has trivial solution: take

$$x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

However, we are interested in non-trivial vector x.

Define $A_\lambda := A - \lambda I$ new matrix

Then $A_\lambda x = 0$ for $x \neq 0$ means A_λ matrix must be singular (singular matrix has determinant zero).

For a number α, we know what it means to say $\alpha \neq 0$

But what about $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ where $x_1, x_2, \ldots, x_n$ are numbers?

we are saying

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

$\Downarrow$

This would be true if atleast one x_i is not equal to zero, ie,

there is i, $i = 1, 2, \ldots, n$, s.t $x_i \neq 0$

then $x \neq 0$

Therefore, the condition is

(3) $A_\lambda x = 0$

(4) $\det(A_\lambda) = 0$

We first use equation (4) to compute λ as it has only one unknown λ.

And then use (3) to compute x once we have λ.

(A) Determinant problem

we have to find λ such that

(5) $\boxed{\det(A_\lambda) = \det(A - \lambda I) = 0}$

$\downarrow$

This is called characteristics equation or characteristics polynomial. Polynomial because equation (5) turns out to be n^{th} order polynomial equation for λ (n^{th} order for matrix $A_{n \times n}$). Therefore, (5) has at most n unique roots or eigenvalues.

Example : $A_{2\times 2} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Then

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix}\right)$$

$$\Rightarrow \det\left(\begin{bmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0$$

$\Rightarrow (2-\lambda)(1-\lambda) = 0$ quadratic equation

$$\Rightarrow \boxed{\lambda = 1, 2}$$

For a general $n\times n$ matrix $A - \lambda I$ is simply

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Some $n\times n$ matrix examples

(1) A is identity matrix. Then

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & \cdots & 0 \\ 0 & 1-\lambda & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1-\lambda \end{bmatrix}$$

$\therefore \quad \det(A - \lambda I) = (1-\lambda)^n = 0$

$\Rightarrow \lambda = 1, 1, \ldots, 1 \quad \leftarrow$ n repeated roots

→ Identity matrix $I_{n\times n}$ has only one unique eigenvalue $\lambda = 1$

→ For identity matrix $I_{n\times n}$, every column vector $x_{n\times 1}$ is eigenvector!

↓

Because for any x, $\quad Ix = x$

$\Rightarrow \quad Ix = \lambda x \quad$ with $\lambda = 1$.

(2) Take a sparse matrix

$$A_{n\times n} = \begin{bmatrix} a & c & 0 & 0 & \cdots & & & 0 \\ b & a & c & 0 & \cdots & & & 0 \\ 0 & b & a & c & \cdots & & & 0 \\ 0 & 0 & b & a & \cdots & & & 0 \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 \cdots & 0 & b & a & c \\ 0 & 0 & 0 & 0 \cdots & 0 & 0 & a & b \end{bmatrix}$$

where a, b, c are three numbers.

When $b = c = 1$, eigenvalues of $A_{n\times n}$ are

$$\lambda_k = a - 2\cos\left(k\pi/(n+1)\right), \quad k = 1, 2, \ldots, n$$

When b and c are general,

$$\lambda_k = a - 2\sqrt{bc}\cos\left(k\pi/(n+1)\right), \quad k = 1, 2, \ldots, n$$

Above is a difficult result. Refer to following journal article

"Eigenvalues of tridiagonal pseudo-Toeplitz matrices"

By Kulkarni, Schmidt, Tsui. Linear Algebra and its Applications $\boxed{1999}$

(B) Solving for eigenvectors Once eigenvalues are computed, we can use equation $(A-\lambda I)x=0$ to solve for x.

Let λ is known. Because $(A-\lambda I)$ is a singular matrix, we will find that n equations will

$$\boxed{\underbrace{(A-\lambda I)}_{}x=0 \quad \Rightarrow \quad \begin{aligned} b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n &= 0 \\ b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n &= 0 \\ &\vdots \\ b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nn}x_n &= 0 \end{aligned}}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

not be enough to solve for n-unknowns in $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

We will encounter either:

(i) two equations out of n-equation is same except a common factor

or

(ii) one of the equation is trivial (e.g., $0=0$).

Therefore, in addition to $(A-\lambda I)x=0$, we supply additional equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \quad \text{where} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

So

$$\boxed{\begin{aligned} & x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \\ \& \quad & (A-\lambda I)x = 0 \end{aligned}}$$

should be sufficient to solve for x, given λ.

Examples

1. for $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, we found $\lambda = 1, 2$.

Consider $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

then

(i) $x_1^2 + x_2^2 = 1$

(ii) $(A - \lambda I)x = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow (2-\lambda)x_1 = 0$ —— (6)

& $x_1 + (1-\lambda)x_2 = 0$ —— (7)

If $\lambda = 1$, then (5) & (6) are

$x_1 = 0$
$x_1 = 0$ } same information

If $\lambda = 2$, then (5) & (6) are

$0 \cdot x_1 = 0$ $\rightarrow$ trivial equation

$x_1 - x_2 = 0$

In both cases, we only have one equation from $(A - \lambda I)x = 0$ that is useful and therefore we require additional condition such as $x_1^2 + x_2^2 = 1$ to fully solve the problem.

For $\lambda = 1$, $x_1 = 0$, & from $x_1^2 + x_2^2 = 1 \Rightarrow x_2 = 1$

so $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is eigenvector

For $\lambda = 2$, $x_1 = x_2$ so from $x_1^2 + x_2^2 = 1$

$$x_2^2 + x_2^2 = 1 \Rightarrow x_2 = \frac{1}{\sqrt{2}} = x_1$$

So $x = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ is eigenvector.

2. Find eigenvalues & eigenvectors for matrix

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

3. Example for $n=3$. Take

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Determinant problem

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left((2-\lambda)^2 - 1\right) - (2-\lambda) = 0$$

$$\Rightarrow (2-\lambda)\left((2-\lambda)^2 - 2\right) = 0$$

$$\Rightarrow 2-\lambda = 0, \quad (2-\lambda)^2 - 2 = 0$$

$\Rightarrow \lambda = 2, \qquad \lambda^2 - 4\lambda + 4 - 2 = 0$

$\Rightarrow \lambda = 2, \qquad \lambda = \dfrac{4 \pm \sqrt{16 - 8}}{2}$

$\Rightarrow \lambda = 2, \qquad \lambda = 2 \pm \sqrt{2}$

$\Rightarrow \boxed{\lambda = 2,\ 2+\sqrt{2},\ 2-\sqrt{2}}$ three unique eigen values.

<u>Eigenvector problem</u> let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$(A - \lambda I)\, x = 0$

$$(2-\lambda)\, x_1 + x_2 = 0 \qquad \text{--- (8)}$$

$$x_1 + (2-\lambda)\, x_2 + x_3 = 0 \qquad \text{--- (9)}$$

$$x_2 + (2-\lambda)\, x_3 = 0 \qquad \text{--- (10)}$$

additional equation $\longrightarrow$

$$x_1^2 + x_2^2 + x_3^2 = 1 \qquad \text{--- (11)}$$

<u>For $\lambda = 2$</u>

two redundant equations

$x_2 = 0$

$x_1 + x_3 = 0$

$x_2 = 0$

$x_1 + x_3 = 0$

$x_2 = 0$

$x_1^2 + x_2^2 + x_3^2 = 1$

three equations three unknowns

$x_1 = -x_3, \quad x_2 = 0$

$\therefore$ from last equation

$x_3^2 + 0 + x_3^2 = 1 \Rightarrow x_3 = \dfrac{1}{\sqrt{2}}$

$\therefore \quad x_1 = -\dfrac{1}{\sqrt{2}}, \quad x_2 = 0, \quad x_3 = \dfrac{1}{\sqrt{2}}$

Thus for $\lambda = 2$, $x = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ is eigenvector.

For $\lambda = 2+\sqrt{2}$

$$\left\{\begin{array}{lll} -\sqrt{2}x_1 + x_2 = 0 & \longrightarrow & x_2 = \sqrt{2}x_1 \\ x_1 - \sqrt{2}x_2 + x_3 = 0 & \longrightarrow & x_2 = \dfrac{x_1+x_3}{\sqrt{2}} \\ x_2 - \sqrt{2}x_3 = 0 & \longrightarrow & x_3 = \dfrac{x_2}{\sqrt{2}} \end{array}\right.$$

$$\longrightarrow x_2 = \frac{x_1 + x_2/\sqrt{2}}{\sqrt{2}}$$

$$\Rightarrow x_2 = \frac{x_1}{\sqrt{2}} + \frac{x_2}{2}$$

$$\Rightarrow \frac{x_2}{2} = \frac{x_1}{\sqrt{2}}$$

$$\Rightarrow x_2 = \sqrt{2}x_1$$

So we have

$$x_3 = \frac{x_2}{\sqrt{2}}, \quad x_2 = \sqrt{2}x_1$$

"only two useful information"

$$x_3 = \frac{x_2}{\sqrt{2}} = \frac{\sqrt{2}x_1}{\sqrt{2}} = x_1$$

So from $x_1^2 + x_2^2 + x_3^2 = 1 \Rightarrow x_1^2 + 2x_1^2 + x_1^2 = 1$

$$\Rightarrow x_1 = \frac{1}{2}$$

So $x_1 = \frac{1}{2}, \quad x_2 = \frac{1}{\sqrt{2}}, \quad x_3 = \frac{1}{2}$

for $\lambda = 2+\sqrt{2}$, $x = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}$ is eigenvector

For $\lambda = 2-\sqrt{2}$

$$\sqrt{2}x_1 + x_2 = 0 \longrightarrow x_2 = -\sqrt{2}x_1$$

$$x_1 + \sqrt{2}x_2 + x_3 = 0 \longrightarrow x_1 + \sqrt{2}(-\sqrt{2}x_1) + x_1 = 0 \rightarrow 0 = 0$$

$$x_2 + \sqrt{2}x_3 = 0 \longrightarrow x_3 = -\frac{1}{\sqrt{2}}x_2 = x_1$$

$\underline{\underline{2}}$ $\quad x_2 = -\sqrt{2}\, x_1$ $\Big\}$ only two useful information

$x_3 = x_1$

from $x_1^2 + x_2^2 + x_3^2 = 1$, we have

$$x_1^2 + 2x_1^2 + x_1^2 = 1 \rightarrow x_1 = \frac{1}{2}$$

$\underline{\text{So}}$

for $\lambda = 2 - \sqrt{2}$, $\quad x = \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix}$ is eigenvector

Collecting

$\lambda = 2$, $\quad x = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ eigen pair 1

$\lambda = 2 + \sqrt{2}$, $\quad x = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}$ eigen pair 2

$\lambda = 2 - \sqrt{2}$, $\quad x = \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix}$ eigen pair 3.

Next, we continue with the discussion of utility of eigenvalues and eigenvectors.

How does eigenvalues and eigenvectors help us ...

(B) Solving system of differential equations using eigenvalues & eigenvectors

Consider system of ODEs for functions

$v = v(t)$, $w = w(t)$,

$$\frac{dv}{dt} = 4v - 5w \qquad \text{——— (12)}$$

$$\frac{dw}{dt} = 2v - 3w \qquad \text{——— (13)}$$

with initial conditions

$v(0) = 8 = v_0$

$w(0) = 5 = w_0$

For ODE (ordinary differential equation) (u is a single function)

$$\frac{du}{dt} = a\,u \quad \text{with } \underbrace{u(0) = u_0}_{\text{initial condition}}$$

where $u = u(t)$ function of t

a = constant

we can solve to get

$$\boxed{u(t) = e^{at} u_0}$$

Suppose, $u = \begin{bmatrix} v \\ w \end{bmatrix}$, i.e. u is a column vector and depends on t because its elements v and w depend on t.

Suppose $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

$$\frac{du}{dt} = \begin{bmatrix} \frac{dv}{dt} \\ \frac{dw}{dt} \end{bmatrix}$$

i.e. derivative of u is obtained by "element-wise" derivative of column vector !!

With above notations, verify that (12) — (13) can be written as

$$\frac{du}{dt} = Au \qquad \text{——— (14)}$$

with initial condition $u(0) = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$.

Therefore, (14) is a matrix representation of equations (12) and (13).

Ques: How do we solve equations (12) and (13) (or equivalently equation (14))?

Ans: We first obtain generic solution of $\frac{du}{dt} = Au$ using eigenvalues & eigenvectors and the obtain complete/specific solution that in addition satisfies the initial condition $u(0) = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$.

Remark 6: If $A_{n \times n}$ has nonzero real eigenvalues, we can find eigenvectors $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ such that any column vector $\underline{y}_{n \times 1}$ can be represented as

$$\underline{y} = \alpha_1 \underline{x}_1 + \cdots + \alpha_n \underline{x}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers and depends on $\underline{y}$.

Above is another way of saying that any column vector is nothing but sum of eigenvectors where each eigenvectors are appropriately scaled by a number $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

This idea will be used to solve (14) using eigenvectors of matrix A.

Solving equation (14)

Motivated from solution of single ODE, suppose

$$v(t) = e^{\lambda t} y \qquad \text{(15)}$$

$$w(t) = e^{\lambda t} z \qquad \text{(16)}$$

where λ, y, z are three numbers to be determined.

So

(17) —— $u = u(t) = e^{\lambda t}\begin{bmatrix} y \\ z \end{bmatrix} \Rightarrow \frac{du}{dt} = \lambda e^{\lambda t}\begin{bmatrix} y \\ z \end{bmatrix} = \lambda u$

but from equation (14),

(18) —— $Au = \frac{du}{dt}$

$\Rightarrow Au = \lambda u$ for all t as u is a function of t

$\Rightarrow A\left(e^{\lambda t}\begin{bmatrix} y \\ z \end{bmatrix}\right) = \lambda e^{\lambda t}\begin{bmatrix} y \\ z \end{bmatrix}$ ($e^{\lambda t}$ is just a number)

$\Rightarrow \cancel{e^{\lambda t}} A\begin{bmatrix} y \\ z \end{bmatrix} = \lambda \cancel{e^{\lambda t}}\begin{bmatrix} y \\ z \end{bmatrix}$

$\Rightarrow A\begin{bmatrix} y \\ z \end{bmatrix} = \lambda\begin{bmatrix} y \\ z \end{bmatrix}$

$\Rightarrow (A - \lambda I)x = 0$ where $x = \begin{bmatrix} y \\ z \end{bmatrix}$

eigenvalue problem.

1. Eigenvalues of A

Recall that

$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\Rightarrow \begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow -(4-\lambda)(3+\lambda) + 10 = 0$

$\Rightarrow -(12 - 3\lambda + 4\lambda - \lambda^2) + 10 = 0$

$\Rightarrow \lambda^2 - \lambda - 2 = 0$

$\Rightarrow \boxed{\lambda = -1, 2}$

2. <u>Eigen vectors of A</u> $\qquad (A-\lambda I)\,x=0$ and $x_1^2+x_2^2=1$

<u>For $\lambda=-1$</u>

$$\begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{array}{l} \rightarrow 5x_1=5x_2 \Rightarrow x_1=x_2 \\ \rightarrow 2x_1=2x_2 \Rightarrow x_1=x_2 \end{array}$$

So

$$x_1^2+x_1^2=1 \Rightarrow x_1=\frac{1}{\sqrt{2}}=x_2$$

eigenvector is $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \Rightarrow \sqrt{2}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is also eigenvector

<u>For $\lambda=2$</u>

$$\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow 2x_1=5x_2 \Rightarrow x_1=\frac{5}{2}x_2$$

So

$$\left(\frac{5}{2}\right)^2 x_2^2+x_2^2=1 \rightarrow x_2^2=\frac{1}{1+\frac{25}{4}}=\frac{4}{29}$$

$$\Rightarrow x_2=\frac{2}{\sqrt{29}}$$

So eigen vector is $\begin{bmatrix} 5/\sqrt{29} \\ 2/\sqrt{29} \end{bmatrix} \rightarrow \sqrt{29}\begin{bmatrix} 5/\sqrt{29} \\ 2/\sqrt{29} \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is also an eigenvector.

<u>Thus</u>

$\lambda=-1$, $x_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigen pair 1

$\lambda=2$, $x_2=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ eigen pair 2

(19) <u>Thus</u> $u_1(t)=e^{\lambda_1 t}x_1=e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $u_2(t)=e^{\lambda_2 t}x_2=e^{2t}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

both u_1 & u_2 solve $\frac{du}{dt} = Au$

Now: since $\frac{du_1}{dt} = Au_1$, $\frac{du_2}{dt} = Au_2$,

we have

$$\frac{d}{dt}(u_1 + u_2) = \overset{= Au_1}{\frac{du_1}{dt}} + \overset{= Au_2}{\frac{du_2}{dt}}$$

$$= Au_1 + Au_2$$

$$\Rightarrow \frac{d}{dt}(u_1 + u_2) = A(u_1 + u_2)$$

In fact given any two numbers α & β,

$$\frac{d}{dt}\underbrace{(\alpha u_1 + \beta u_2)}_{u} = A\underbrace{(\alpha u_1 + \beta u_2)}_{u}$$

$$\Rightarrow \frac{du}{dt} = Au$$

So $u = \alpha u_1 + \beta u_2$ for any possible numbers α, β

solves $\frac{du}{dt} = Au$. $\longrightarrow$ why?

Because, u_1 & u_2

solve $\frac{du}{dt} = Au$.

Idea: Write solution to our original problem

as $u = \alpha u_1 + \beta u_2$

where α and β must be computed

to satisfy the original problem.

Original problem

$\frac{du}{dt} = Au$

$u(0) = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$

Well, if $u = \alpha u_1 + \beta u_2$ then $\frac{du}{dt} = Au$ ✓

& $u(0) = \alpha u_1(0) + \beta u_2(0) = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$ two given numbers

Note, from (19)

$u_1(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $u_2(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ as $e^0 = 1$

So $u(0) = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \alpha + 5\beta \\ \alpha + 2\beta \end{bmatrix}$

So find α, β such that

$$\begin{bmatrix} \alpha + 5\beta \\ \alpha + 2\beta \end{bmatrix} = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} \Rightarrow \left.\begin{array}{l} \alpha + 5\beta = v_0 \\ \alpha + 2\beta = w_0 \end{array}\right\} \begin{array}{l} \text{2 equations} \\ \text{2 unknowns} \end{array}$$

Once we have α, β, the complete solution is

$u(t) = \alpha u_1(t) + \beta u_2(t)$

and $u(t)$ satisfies (i) $\frac{du}{dt} = Au$

(ii) $u(0) = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$

So we used eigenvalue problem to

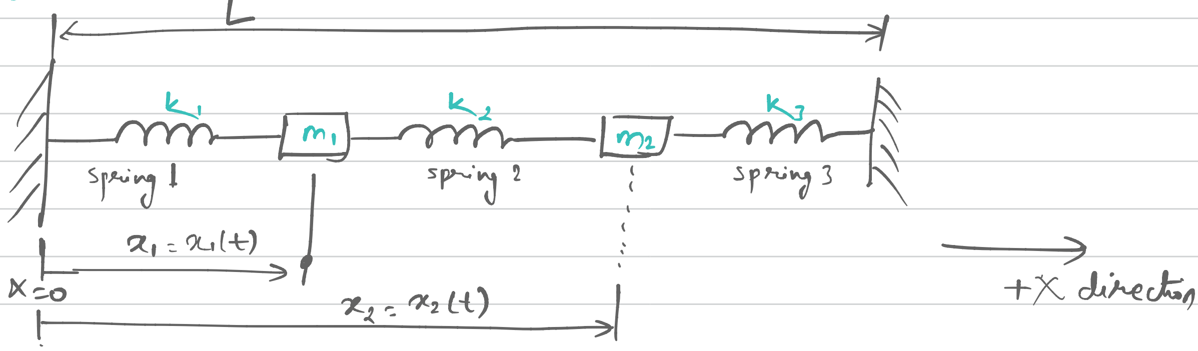
(i) Convert $\frac{du}{dt} = Au$ into $Au = \lambda u$ problem

(ii) Solved for all pairs (λ, u) say $(\lambda_1, u_1), (\lambda_2, u_2), \ldots, (\lambda_n, u_n)$

(iii) wrote solution as $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \Rightarrow \frac{du}{dt} = Au$ automatically

(iv) found $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that initial condition is satisfied.

(c) Using eigenvalue problem to solve another ODE problem



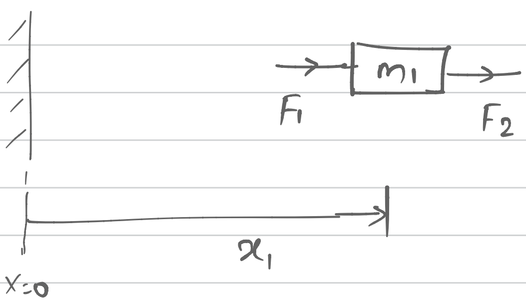
Let $x_1 = x_1(t)$ and $x_2 = x_2(t)$ denote the position of mass from left wall and let L_1, L_2, L_3 are equilibrium length of three springs. L is fixed and we assume $L = L_1 + L_2 + L_3$.

Assuming no damping, for both masses, we have following balance of linear momentum

rate of change in linear momentum

= external forces

mass m_1 free body diagram



F_1 = force by spring 1

F_2 = force by spring 2.

for spring 1, current length is $l_1 = x_1$

change in length $\delta_1 = l_1 - L_1$

$= x_1 - L_1$

So $F_1 = -k_1 \delta_1$ (opposite to δ_1)

$= -k_1(x_1 - L_1)$

Similarly, for spring 2, current length $l_2 = x_2 - x_1$

change in length $\delta_2 = l_2 - L_2$

In this case

if $\delta_2 > 0$ then F_2 is in +ve x-direction

if $\delta_2 < 0$ then F_2 is in -ve x-direction

So $F_2 = k_2 \delta_2$ satisfies both observations.

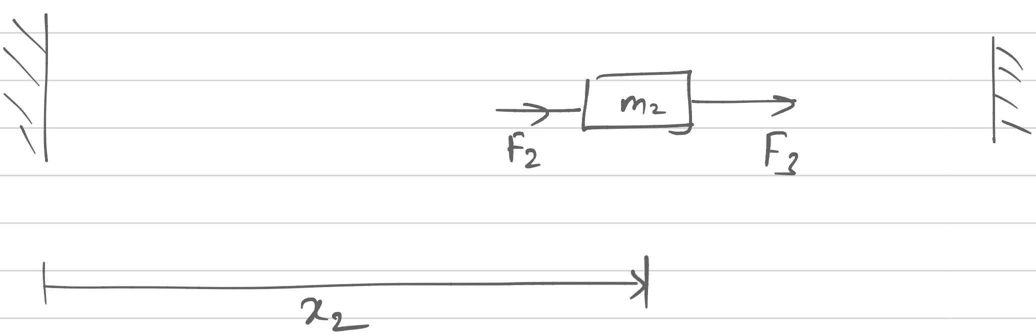
So

$$\frac{d}{dt}\left(m_1 \frac{dx_1}{dt}\right) = F_1 + F_2$$

$$\Rightarrow m_1 \frac{d^2 x_1}{dt^2} = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_2)$$

$$\Rightarrow \boxed{m_1 \frac{d^2 x_1}{dt^2} = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_2)}$$

mass m_2



Spring 2 : $\delta_2 = l_2 - L_2 = x_2 - x_1 - L_2$

if $\delta_2 > 0$, F_2 is $-$ve

if $\delta_2 < 0$, F_2 is $+$ve

So

$$F_2 = -k_2 \delta_2 \quad \text{(opposite sign of } \delta_2\text{)}$$

Spring 3 : Current length $l_3 = L - x_2$ (L is the fixed distance between two walls)

$L = L_1 + L_2 + L_3$

So change in length

$$\delta_3 = l_3 - L_3$$

$$= L - x_2 - L_3$$

$$= L_1 + L_2 + \cancel{L_3} - x_2 - \cancel{L_3}$$

$$= L_1 + L_2 - x_2$$

if $\delta_3 > 0$, F_3 is $+$ve

if $\delta_3 < 0$, F_3 is $-$ve

So

$$F_3 = k_3 \delta_3$$

Then

$$m_2 \frac{d^2 x_2}{dt^2} = F_2 + F_3 = -k_2(x_2 - x_1 - L_2) + k_3(L_1 + L_2 - x_2)$$

$$\Rightarrow \boxed{m_2 \frac{d^2 x_2}{dt^2} = -k_2(x_2 - x_1 - L_2) + k_3(L_1 + L_2 - x_2)}$$

Thus, we have (L_1, L_2, L_3 are equilibrium lengths of three springs)

(20) — $$m_1 \frac{d^2x_1}{dt^2} = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_2)$$

(21) — $$m_2 \frac{d^2x_2}{dt^2} = k_3(L_1 + L_2 - x_2) - k_2(x_2 - x_1 - L_2)$$

let

$$y_1 = x_1 - L_1 \Rightarrow \frac{d^2y_1}{dt^2} = \frac{d^2x_1}{dt^2}$$

$$y_2 = x_2 - (L_1 + L_2) \Rightarrow \frac{d^2y_2}{dt^2} = \frac{d^2x_2}{dt^2}$$

why?

$$x_2 - x_1 - L_2 = y_2 + \cancel{L_1} + \cancel{L_2} - y_1 - \cancel{L_1} - \cancel{L_2} = y_2 - y_1$$

Then from (20) & (21)

(22) — $$\boxed{m_1 \frac{d^2y_1}{dt^2} = -k_1 y_1 + k_2(y_2 - y_1)}$$

(23) — $$\boxed{m_2 \frac{d^2y_2}{dt^2} = -k_3 y_2 - k_2(y_2 - y_1)}$$

Equations (22) & (23) are much nicer compared to (20) & (21).

If we have y_1 & y_2 from (22) & (23), then we can find the location of mass m_1 & m_2 from left wall by using

(24) — $$\boxed{x_1 = y_1 + L_1}$$

(25) — $$\boxed{x_2 = y_2 + (L_1 + L_2)}$$

Initial conditions for (22) & (23)

we let (i) $y_1(0) = 0$, $y_2(0) = 0$ — (26)

(ii) $\dot{y}_1(0) = v_0$, $\dot{y}_2(0) = w_0$. — (27)

Now, we apply eigenvalue method to solve (22) & (23).

Note first that, for 2nd order ODE (only one equation for $u = u(t)$)

$$\frac{d^2u}{dt^2} = -au, \quad u(0) = 0, \ \dot{u}(0) = \dot{u}_0$$

the solution is $u(t) = \alpha \sin(\beta t)$

where $\alpha = \frac{\dot{u}_0}{\sqrt{a}}$, $\beta = \sqrt{a}$.

take $u = \alpha \sin(\beta t)$

then $\frac{d^2u}{dt^2} = -\beta^2 u$

$a = \beta^2 \Rightarrow \beta = \sqrt{a}$.

& $u(0) = 0$

$\dot{u}(0) = \alpha\beta = \dot{u}_0$

$\Rightarrow \alpha\sqrt{a} = \dot{u}_0$

$\Rightarrow \alpha = \frac{\dot{u}_0}{\sqrt{a}}$

So $\boxed{u = \frac{\dot{u}_0}{\sqrt{a}} \sin(\sqrt{a}\, t)}$

Check

(i) $\frac{du}{dt} = \alpha\beta\cos(\beta t)$

$\frac{d^2u}{dt^2} = -\alpha\beta^2\sin(\beta t) = -\beta^2 u$

but $\beta = \sqrt{a} \Rightarrow \frac{d^2u}{dt^2} = -au$ ✓

(ii) $u(0) = 0$ ✓

(iii) $\dot{u}(0) = \alpha\beta\cos(0) = \alpha\beta = \frac{\dot{u}_0}{\cancel{\sqrt{a}}}\cancel{\sqrt{a}} = \dot{u}_0$ ✓

Let

$$u(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} -k_1/m_1 - k_2/m_1 & k_2/m_1 \\ k_2/m_2 & -k_2/m_2 - k_3/m_2 \end{bmatrix}$$

then equations (22) & (23) can be written as

(28) — $$\frac{d^2u}{dt^2} = Au$$

and

initial conditions (i) $u(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ — (29)

(ii) $\dot{u}(0) = \begin{bmatrix} \dot{y}_1(0) \\ \dot{y}_2(0) \end{bmatrix} = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$ — (30)

Motivated by generic solution of 2^{nd} order ODE of single function ($\alpha \sin(\beta t)$), we assume

$$u(t) = \sin(\omega t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{——— (31)}$$

Then

$$\frac{d^2u}{dt^2} = -\omega^2 \sin(\omega t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

$$Au = A\left(\sin(\omega t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \sin(\omega t)\, A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So from

$$\frac{d^2u}{dt^2} = Au$$

$$\Rightarrow -\omega^2 \cancel{\sin(\omega t)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cancel{\sin(\omega t)}\, A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow (A + \underbrace{\omega^2}_{-\lambda} I) \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{(A - \lambda I)\, x = 0}$$

where $\lambda = -\omega^2$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$A = \begin{bmatrix} -k_1/m_1 - k_2/m_1 & k_2/m_1 \\ k_2/m_2 & -k_2/m_2 - k_3/m_3 \end{bmatrix}$$

step 1: Solve $(A - \lambda I)\, x = 0$

to get two pairs of (λ_1, x_1), (λ_2, x_2)

$\lambda_1 \to -\omega_1^2$, $x_1 \to$ column vector; $\lambda_2 \to -\omega_2^2$, $x_2 \to$ column vector

step 2: eigen solutions

$$u_1 = u_1(t) = \sin(\omega_1 t)\, x_1, \qquad u_2 = u_2(t) = \sin(\omega_2 t)\, x_2$$

step 3: since u_1 & u_2 satisfy $\frac{d^2u}{dt^2} = Au$ (Check!)

$$u = \alpha u_1 + \beta u_2$$

also satisfies $\frac{d^2u}{dt^2} = Au$

Step 4: let $u = \alpha u_1 + \beta u_2$ then trivialy $\frac{d^2u}{dt^2} = Au$

Find numbers, α, β, such that

$$\underbrace{u(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{trivially satisfied due to } \sin(wt) \text{ function}} \quad \text{and} \quad \underbrace{\dot{u}(0) = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}}_{\text{two equations for two unknowns.}}$$

So we can solve for α & β.

Example: Take $k_1 = k_2 = k_3 = k$, $m_1 = m_2 = 1$

then

$$A = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix} -2k-\lambda & k \\ k & -2\lambda-\lambda \end{bmatrix}$$

Eigenvalues

$$A \quad |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -2k-\lambda & k \\ k & -2k-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2k+\lambda)^2 - k^2 = 0$$

$$\Rightarrow \lambda^2 + 4k\lambda + 4k^2 - k^2 = 0$$

$$\Rightarrow \lambda = \frac{-4k \pm \sqrt{16k^2 - 12k^2}}{2}$$

$\Rightarrow \lambda = -2k \pm k$

$\Rightarrow \lambda = -k, -3k$

Eigenvectors

$\lambda = -k$,

$$\begin{bmatrix} -k & k \\ k & -k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2$$

$\&$ $x_1^2 + x_2^2 = 1$

$\Rightarrow x_1 = \frac{1}{\sqrt{2}} = x_2$

$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ is eigenvectors $\rightarrow$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is also eigenvectors (why?)

$\lambda = -3k$,

$$\begin{bmatrix} k & k \\ k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -x_2$$

So $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ is eigenvectors $\longrightarrow$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is also eigenvectors

So

$\lambda_1 = -k$, $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigen pair 1

$\lambda_2 = -3k$, $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ eigen pair 2

Thus $\lambda = -\omega^2 \Rightarrow \omega^2 = -\lambda$ So if $\lambda = -k \Rightarrow \omega = \sqrt{k}$

$\lambda = -3k \Rightarrow \omega = \sqrt{3k}$

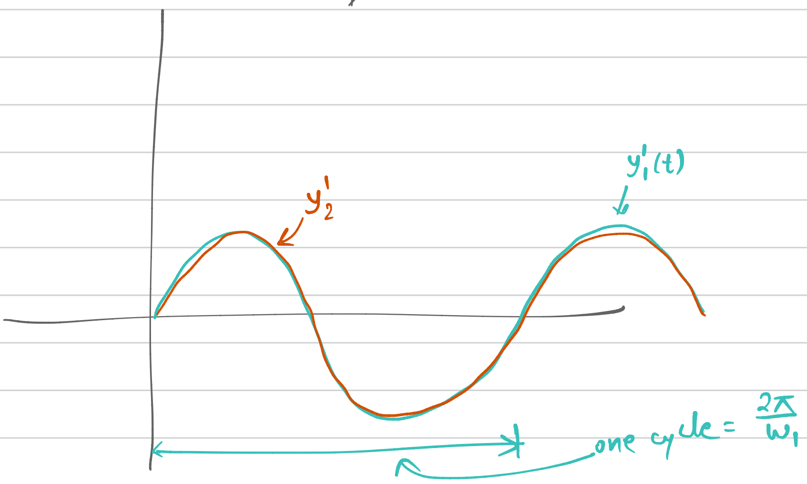
we have $u = u(t) = \sin(\omega t)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

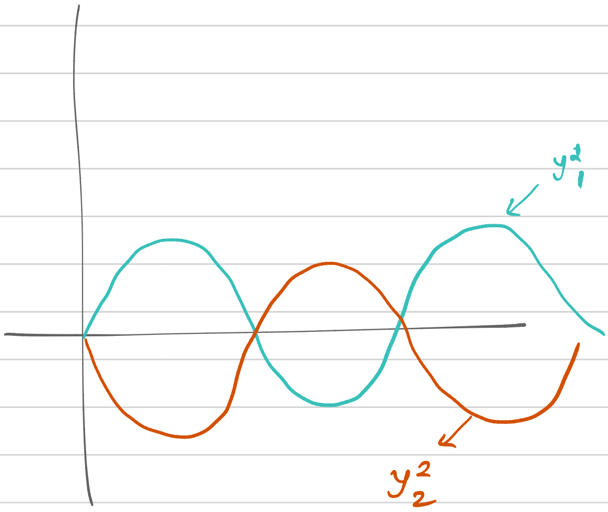
$$u_1 = \sin(\omega_1 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad u_2 = \sin(\omega_2 t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\downarrow$

$$u_1 = \begin{bmatrix} y_1^1(t) \\ y_2^1(t) \end{bmatrix} = \begin{bmatrix} \sin(\omega_1 t) \\ \sin(\omega_1 t) \end{bmatrix}, \qquad u_2 = \begin{bmatrix} y_1^2 \\ y_2^2 \end{bmatrix} = \begin{bmatrix} \sin(\omega_2 t) \\ -\sin(\omega_2 t) \end{bmatrix}$$

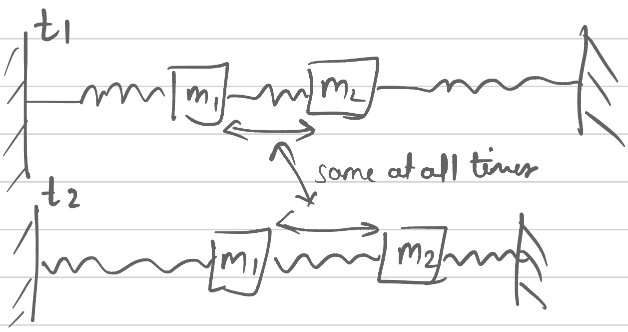
$\Downarrow$

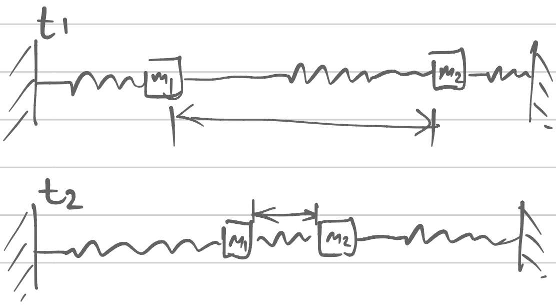




Both mass are vibrating in same direction

mass are vibrating in opposite direction





So

- eigenvalue determines the wavelength $\left(\frac{2\pi}{\omega}\right)$ or frequency (ω)
- eigenvectors determine the relative motion of mass and amplitude

for $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\longrightarrow$ both mass are vibrating in same direction

for $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\longrightarrow$ vibrating in opposite direction.

Combined, eigenvalues & eigenvectors allow us to solve system of ODEs and also linear system of equations.

For further readings, read the following book

"Linear Algebra and Its Applications"

by Gilbert Strang