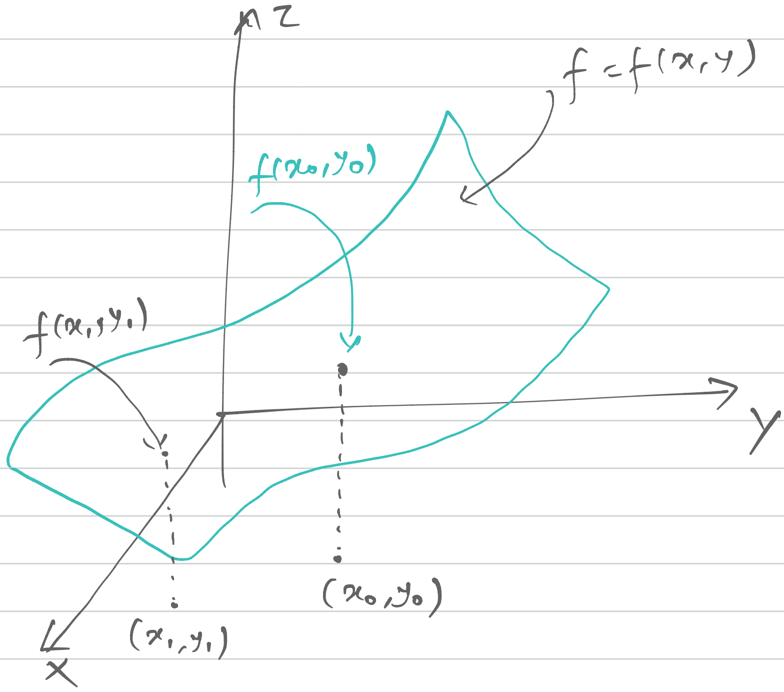
Lecture 19

Solving nonlinear system of equations

$$f = f(x,y)$$



$$\boxed{\begin{aligned} f(x_1,y_1) &\approx f(x_0,y_0) \\ &+ \frac{\partial f}{\partial x}(x_0,y_0)\,(x_1-x_0) \\ &+ \frac{\partial f}{\partial y}(x_0,y_0)\,(y_1-y_0) \end{aligned}}$$

$$\Downarrow$$

$$= f(x_0,y_0) + \underbrace{\left[\frac{\partial f}{\partial x}(x_0,y_0),\ \frac{\partial f}{\partial y}(x_0,y_0)\right]}_{\text{row vector} \;\to\; J_f(x^0)} \underbrace{\begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}}_{\text{column vector}}$$

$$\text{column vector} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$= f(x_0,y_0) + J_f(x^0)\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - J_f(x^0)\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}, \quad x^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} \; ?$$

$$\boxed{\begin{aligned} f_1(x_1, x_2) &= 0 \\ f_2(x_1, x_2) &= 0 \end{aligned}}$$

$f(x, y)$

$f(a), \quad a = \begin{bmatrix} x \\ y \end{bmatrix}$

Ideally,

$$f_1(x^1) = 0$$

$$f_2(x^1) = 0$$

$$0 = f_1(x^1) \approx f_1(x^0) + J_{f_1}(x^0)\, x^1 - J_{f_1}(x^0)\, x^0 = 0$$

$$0 = f_2(x^1) \approx f_2(x^0) + J_{f_2}(x^0)\, x^1 - J_{f_2}(x^0)\, x^0 = 0$$

$$J_{f_1}(x^0)\, x^1 = -f_1(x^0) + J_{f_1}(x^0)\, x^0$$

$$J_{f_2}(x^0)\, x^1 = -f_2(x^0) + J_{f_2}(x^0)\, x^0$$

combine

$a \times b = c$

$1 \times n \quad n \times 1 \quad 1 \times 1$

$$J x^1 = b$$

$$J(x^0) = J = \begin{bmatrix} \text{---} \; J_{f_1}(x^0) \; \text{---} \\ \text{---} \; J_{f_2}(x^0) \; \text{---} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^0) & \frac{\partial f_1}{\partial x_2}(x^0) \\ \frac{\partial f_2}{\partial x_1}(x^0) & \frac{\partial f_2}{\partial x_2}(x^0) \end{bmatrix}$$

$$b = -\begin{bmatrix} f_1(x^0) \\ f_2(x^0) \end{bmatrix} + J(x^0)\, x^0$$

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}, \qquad x^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_n^1 \end{bmatrix}$$

$$\boxed{\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}}$$

$$J(x^{i-1})\, x^i = b(x^{i-1})$$

$$\boxed{\underbrace{J(x^0)}_{\text{matrix}}\, x^1 = \underbrace{b(x^0)}_{\text{column vector}}}$$

$$J(x^0) = \begin{bmatrix} \text{———} & J_{f_1}(x^0) & \text{———} \\ \text{———} & J_{f_2}(x^0) & \text{———} \\ & \vdots & \\ \text{———} & J_{f_n}(x^0) & \text{———} \end{bmatrix}$$

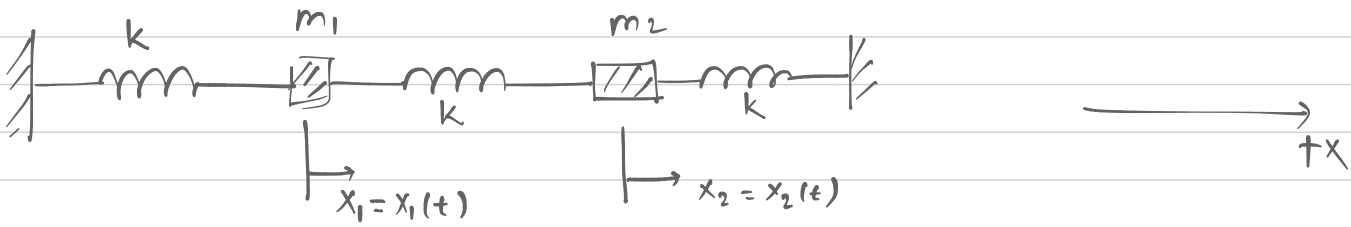
$$J_{f_1}(x^0) = \left[\frac{\partial f_1}{\partial x_1}(x_0), \ \frac{\partial f_1}{\partial x_2}(x^0), \ \ldots, \ \frac{\partial f_1}{\partial x_n}(x^0) \right]$$

$$J_{f_n}(x^0) = \left[\frac{\partial f_n}{\partial x_1}(x^0), \ \frac{\partial f_n}{\partial x_2}(x^0), \ \ldots, \ \frac{\partial f_n}{\partial x_n}(x^0) \right]$$

$$b = -\begin{bmatrix} f_1(x^0) \\ f_2(x^0) \\ \vdots \\ f_n(x^0) \end{bmatrix} + J(x^0)\, x^0$$

$\longrightarrow$ x^0 ✓, x^1 ✓, $J(x^1)\, x^2 = b(x^1)$, x^2 ✓, ...

Eigenvalues and eigenvectors.



$\delta = x_2 - x_1$

$$m_1 \frac{d^2 x_1(t)}{dt^2} = -k x_1 - k|\delta|$$

$$= -k x_1 - k(-(x_2 - x_1))$$

if $X_1 \longrightarrow X_{1new}$

$X_{1new} > X_1$

$\delta_{new} < \delta$

(1) $$\boxed{m_1 \frac{d^2 x_1(t)}{dt^2} = -k x_1 + k(x_2 - x_1)}$$

(2) $$\boxed{m_2 \frac{d^2 x_2(t)}{dt^2} = -k x_2 - k(x_2 - x_1)}$$

$$\frac{d^2 x}{dt^2} = -C x$$

$$\boxed{x = a \sin(bt)}$$

$$\frac{d^2 x}{dt^2} = -a b^2 \sin(bt)$$

$$= -b^2 x$$

if I choose $b^2 = C$

then $$\boxed{\frac{d^2 x}{dt^2} = -C x}$$

we assume

$x_1 = X_1 \sin(\omega t)$

$x_2 = X_2 \sin(\omega t)$

X_1, X_2, ω are numbers and unknown

From (1)

$$-\omega^2 x_1 = -\frac{k}{m_1} x_1 + \frac{k}{m_1}(x_2 - x_1)$$

$$\Rightarrow \boxed{\left(\frac{2k}{m} - \omega^2\right) x_1 - \frac{k}{m_1} x_2 = 0}$$

$$\Rightarrow \left(\left(\frac{2k}{m} - \omega^2\right) X_1 - \frac{k}{m_1} X_2\right) \sin(\omega t) = 0$$

$$\Rightarrow \boxed{\left(\frac{2k}{m} - \omega^2\right) X_1 - \frac{k}{m_1} X_2 = 0} \quad (3)$$

from eq^n 2

$$-\frac{k}{m_2} X_1 + \left(\frac{2k}{m_2} - \omega^2\right) X_2 = 0 \qquad (4)$$

ω^2, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} \frac{2k}{m_1} - \omega^2 & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{2k}{m_2} - \omega^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \boxed{\begin{bmatrix} 2k/m_1 & -k/m_1 \\ -k/m_2 & 2k/m_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \omega^2 \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}$$

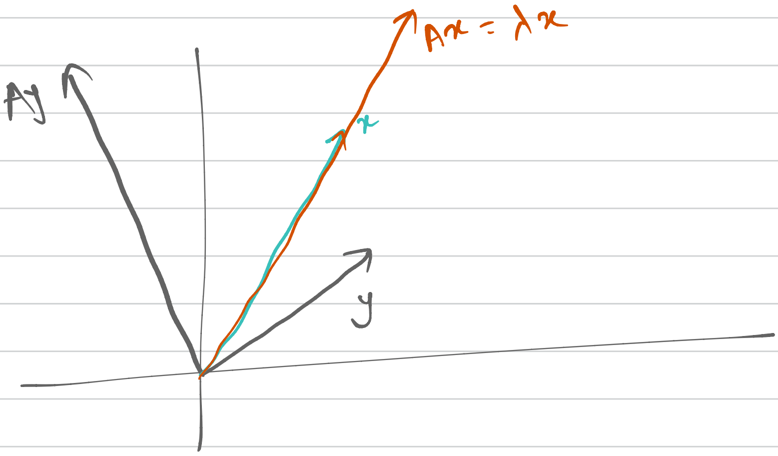
$$\Downarrow$$

$$\boxed{A x = \lambda x}$$

eigenvalue and eigenvector problem

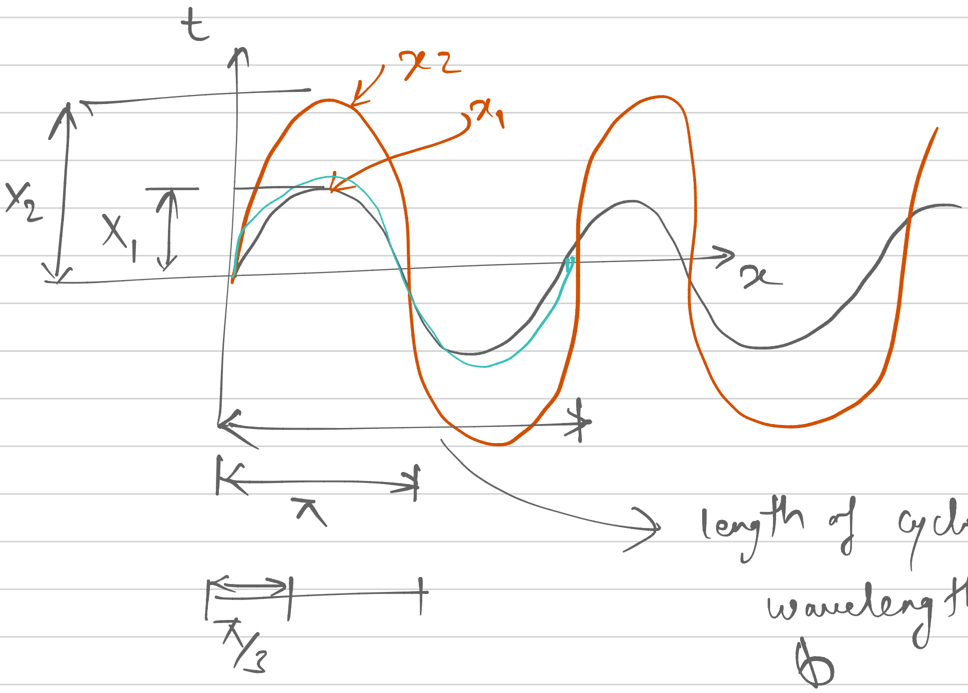
eigenvalue

eigenvector



$A_{2\times 2}$, $x_{2\times 1}$

$x_1 = X_1 \sin(\omega t)$ $x_2 = X_2 \sin(\omega t)$



$\omega = 1$

$\omega = 3$

$\dfrac{2\pi}{\omega}$

length of cycle

wavelength

depends on ω

- $\omega = 1$, wavelength $= 2\pi$
- $\omega = N$, $\dfrac{2\pi}{N}$
- ω, $\dfrac{2\pi}{\omega}$