Lecture 18

- Iterative (numerical) methods
- Solving system of <u>nonlinear equations</u>

- Iterative method

$$
\begin{aligned}
&(1) \quad a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 \\
&(2) \quad a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2 \\
&(3) \quad a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3
\end{aligned}
$$

$$
\boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad x^i = \begin{bmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_n^i \end{bmatrix}}
$$

$\Rightarrow$ $x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{bmatrix}$ as initial guess for a $Ax = b$ problem

$$
x^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \end{bmatrix}
$$

$$
(1) \quad a_{11} x_1^1 + a_{12} \underset{\substack{\downarrow \\ x_2^0}}{x_2^1} + a_{13} \underset{\substack{\downarrow \\ x_3^0}}{x_3^1} = b_1
$$

$$
x_1^1 = \frac{b_1 - a_{12} x_2^0 - a_{13} x_3^0}{a_{11}}
$$

(2) $a_{21}\, x_1^1 + a_{22}\, x_2^1 + a_{23}\, x_3^1 = b_2$

$x_1^1 \to x_1^0$, $x_3^1 \to x_3^0$ Jacobi iteration

x_1^1, x_3^0 Gauss-seidel iteration

$$x_2^1 = \frac{b_2 - a_{21}\, \overset{x_1^1 \uparrow}{x_1^0} - a_{23}\, x_3^0}{a_{22}}$$

(3) $a_{31}\, x_1^1 + a_{32}\, x_2^1 + a_{33}\, x_3^1 = b_3$

$x_1^1 \to x_1^1$, $x_2^1 \to x_2^1$ (Gauss-Seidel iteration)

x_1^0, x_2^0 (Jacobi iteration)

$$x_3^1 = b_3 - a_{31}\, \overset{x_1^1 \uparrow}{x_1^0} - a_{32}\, \overset{x_2^1 \uparrow}{x_2^0}$$

$$x^0 \longrightarrow x^1 \longrightarrow x^2 \longrightarrow \cdots \longrightarrow x^i \longrightarrow x^{i+1}$$

For general $n \times n$ system : let x^i is a solution at i^{th} iteration and want to compute x^{i+1}

$$x_1^{i+1} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}\, x_2^i + a_{13}\, x_3^i + \cdots + a_{1n}\, x_n^i \right) \right]$$

$$x_2^{i+1} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}\, \overset{x_1^{i+1} \uparrow}{x_1^i} + a_{23}\, x_3^i + \cdots + a_{2n}\, x_n^i \right) \right]$$

$\vdots$

$$x_k^{i+1} = \frac{1}{a_{kk}}\Big[b_k - \big(a_{k1} \overset{x_1^{i+1}}{\overset{\uparrow}{x_1^i}} + a_{k2} \overset{x_2^{i+1}}{\overset{\uparrow}{x_2^i}} + \cdots + a_{k(k-1)} \overset{x_{k-1}^{i+1}}{\overset{\uparrow}{x_{k-1}^i}}$$

$$+ a_{k(k+1)} x_{k+1}^i + \cdots + a_{kn} x_n^i \big) \Big]$$

$$\bar{b} = \begin{bmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \vdots \\ \frac{b_n}{a_{nn}} \end{bmatrix} , \quad \bar{A} = \begin{bmatrix} 0 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \frac{a_{23}}{a_{22}} & \cdots & \frac{a_{2n}}{a_{22}} \\ \vdots \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \frac{a_{n3}}{a_{nn}} & \cdots & 0 \end{bmatrix}$$

Write Jacobi iteration

$$\boxed{x^{i+1} = \bar{b} - \bar{A} x^i}$$

$$x^1 = \bar{b} - \bar{A} x^0$$
$$x^2 = \bar{b} - \bar{A} x^1$$
$$\vdots$$

$$\begin{bmatrix} x_1^{i+1} \\ x_2^{i+1} \\ \vdots \\ x_n^{i+1} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_n \end{bmatrix} - \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\ \vdots \\ \bar{a}_{n1} & \bar{a}_{n2} & \cdots & \bar{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_n^i \end{bmatrix}$$

$$x_1^{i+1} = \bar{b}_1 - \left(\bar{a}_{11} x_1^i + \bar{a}_{12} x_2^i + \cdots + \bar{a}_{1n} x_n^i \right)$$

• Relaxation method

Given x^i solution at i^{th} iteration

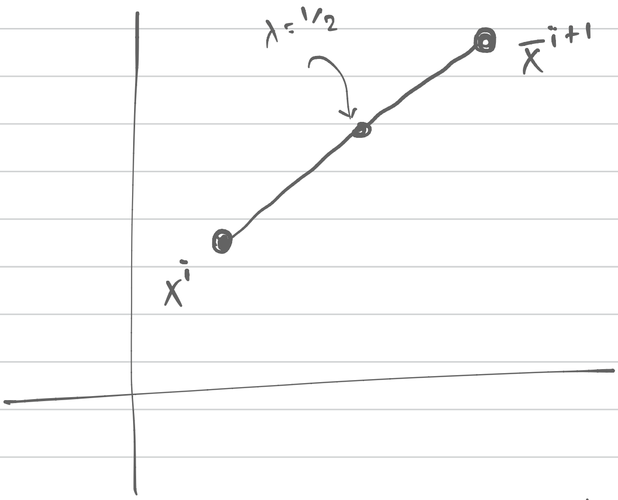
$\Downarrow$

$\bar{x}^{i+1}$ solution using Gauss-Seidel / Jacobi iteration

$$x^{i+1} = \lambda \bar{x}^{i+1} + (1-\lambda) x^i$$

$\lambda = 1 \implies$ get usual iterative method

but if $0 < \lambda < 1 \longrightarrow$ "under relaxation method"



$\lambda > 1 \longrightarrow x^{i+1} = \lambda \bar{x}^{i+1} + (1-\lambda) x^i$

$\downarrow$

"over-relaxation"

- System of nonlinear equations

$a_{11} = x_1 x_2$, $a_{12} = 1$

$a_{21} = x_1^2$, $a_{22} = x_1^2 x_2$

2 unknowns, 2 equations

$f_1(x_1, x_2) = 0$

$f_2(x_1, x_2) = 0$

$f_1 = f_1(x_1, x_2) \leftarrow \boxed{x_1^2 x_2 + x_2 - b_1} = 0$

$f_2 = f_2(x_1, x_2) \leftarrow \boxed{x_1^3 + x_2^2 x_1^2 - b_2} = 0$

It is a multivariable & multi-function roots problem.

x, y are numbers such that

$$f(x,y) = x^2 + y^2 - 2xy + \cos(x)\sin(x) = 0$$

Newton-Raphson for one variable & one function

roots problem: $\Rightarrow f(x) = 0$

x_0 given, x_1

$f(x_1) = 0$

$\Downarrow$

$$\boxed{f(x_0) + \frac{df}{dx}(x_0)(x_1 - x_0)} = 0$$

$$\Rightarrow \boxed{x_1 = x_0 - \frac{f(x_0)}{\frac{df}{dx}(x_0)}}$$

$$X^0 = \begin{bmatrix} x_1^0 \\ x_1^0 \end{bmatrix} \longrightarrow x^1 = \begin{bmatrix} x_1^1 \\ x_1^1 \end{bmatrix} ?$$

solving $Ax = b$

$$f_1(x_1^1, x_2^1) = 0$$

$\Downarrow$ (Taylor's series)

$$\Rightarrow f_1(x_1^0, x_2^0) + \frac{\partial f_1}{\partial x_1}(x_1^0, x_2^0)(x_1^1 - x_1^0) + \frac{\partial f}{\partial x_2}(x_1^0, x_2^0)(x_2^1 - x_2^0) = 0$$

$$\Downarrow$$

$$a_{11} x_1^1 + a_{12} x_2^1 = b_1$$

$$f_2(x_1^1, x_2^1) = 0$$

$$\Downarrow$$

$$\Rightarrow f_2(x_1^0, x_1^0) + \frac{\partial f_2}{\partial x_1}(x_1^0, x_2^0)(x_1^1 - x_1^0) + \frac{\partial f_2}{\partial x_2}(x_1^0, x_2^0)(x_2^1 - x_2^0) = 0$$

$$\Downarrow$$

$$a_{21} x_1^1 + a_{22} x_2^1 = b_2$$

$$a_{11} = \frac{\partial f_1}{\partial x_1}(x_1^0, x_2^0)$$

$$a_{12} = \frac{\partial f_1}{\partial x_2}(x_1^0, x_2^0)$$

$$b_1 = -f_1(x_1^0, x_2^0) + \frac{\partial f_1}{\partial x_1}(x_1^0, x_2^0)\, x_1^0$$

$$\boxed{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$